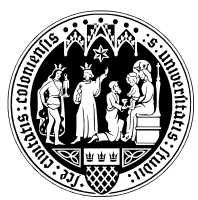
Takum: A New Tapered Precision Machine Number Format

Laslo Hunhold

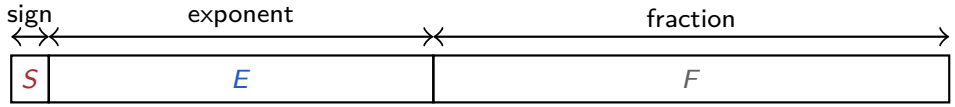
Parallel and Distributed Systems Group
University of Cologne

12th June 2024



IEEE 754 Floating-Point

versus Desirable Design Criteria (1/2)



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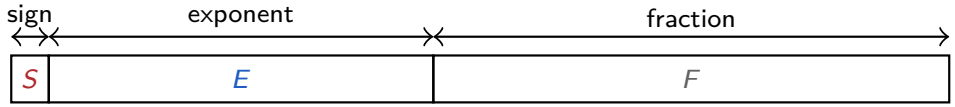

versus Desirable Design Criteria (1/2)

- Represented numbers should reflect those used in computations

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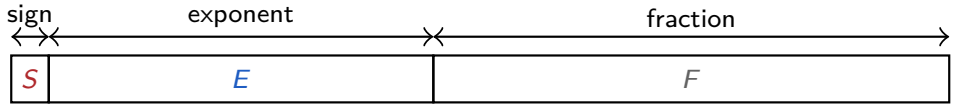

versus Desirable Design Criteria (1/2)

- Represented numbers should reflect those used in computations: Observations

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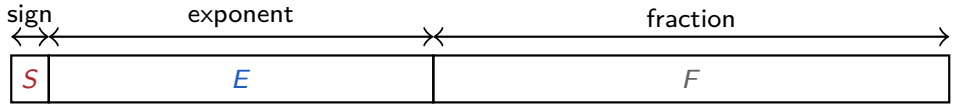

versus Desirable Design Criteria (1/2)

- Represented numbers should reflect those used in computations: Observations:
  - Dynamic Range

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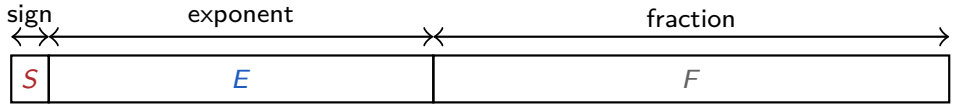

versus Desirable Design Criteria (1/2)

- Represented numbers should reflect those used in computations: Observations:
  - Dynamic Range: General purpose $10^{\pm 55}$ (SI units, constants, etc.)

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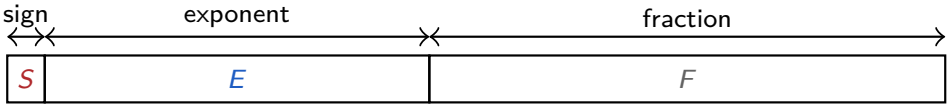

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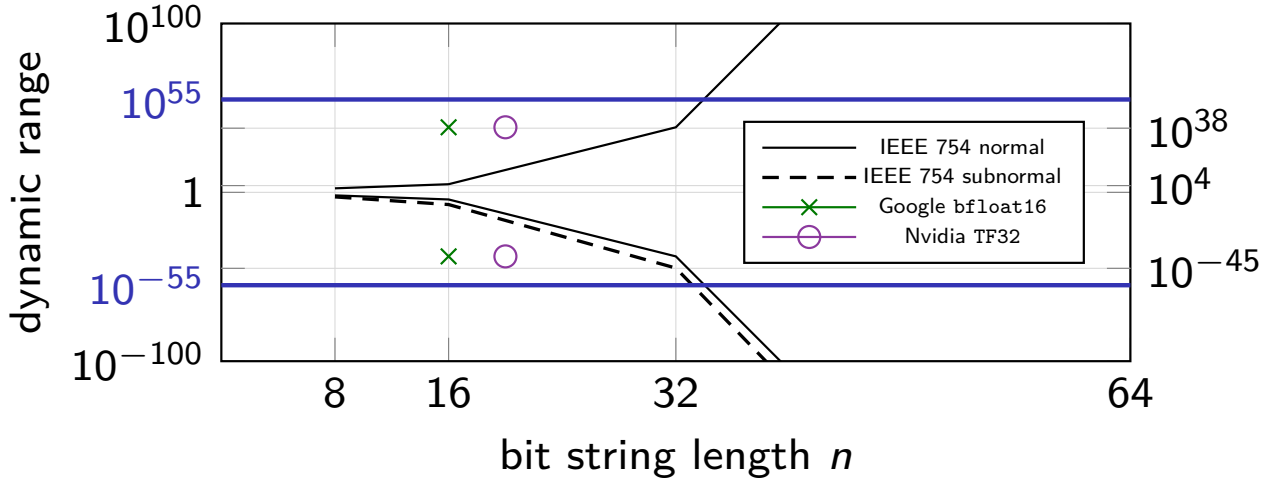

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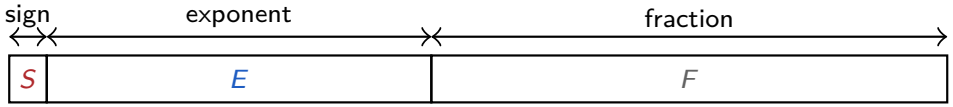

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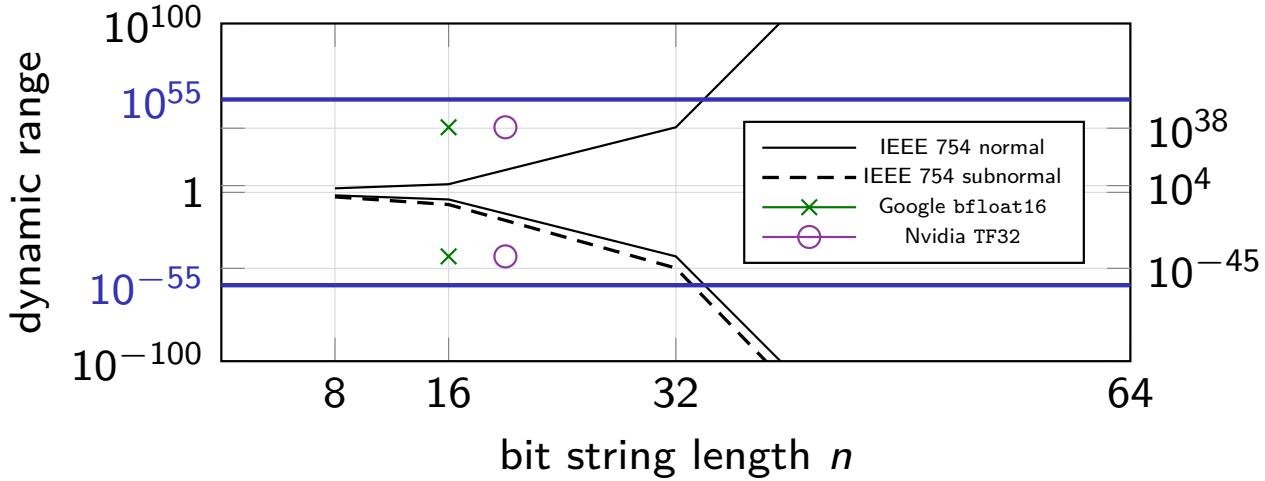

Violated:

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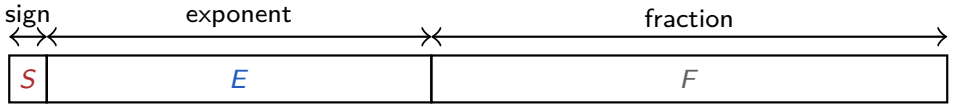

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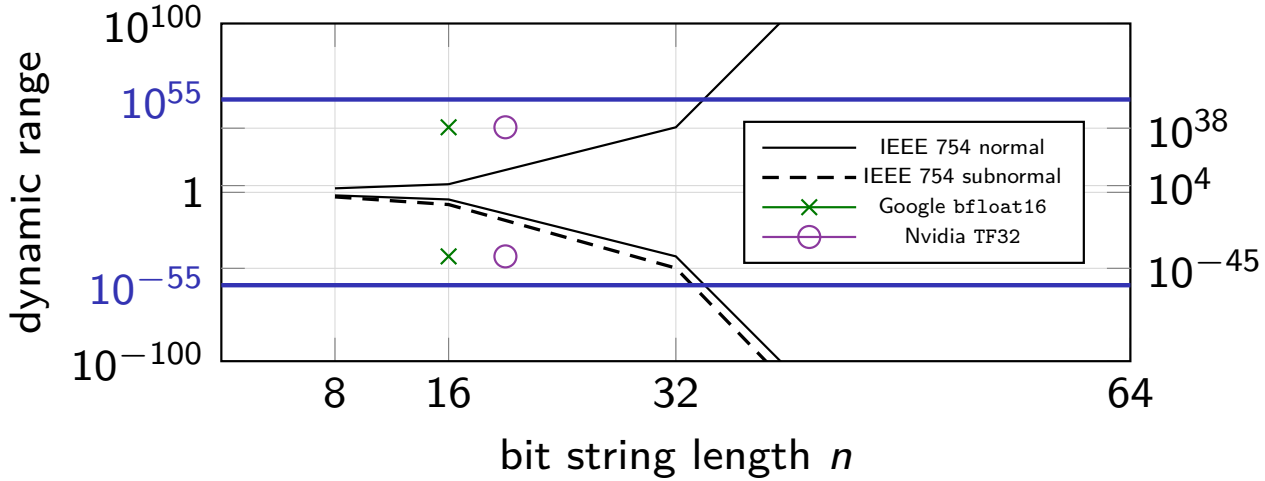

Violated: too small for $n \leq 32$, too large for $n \geq 64$

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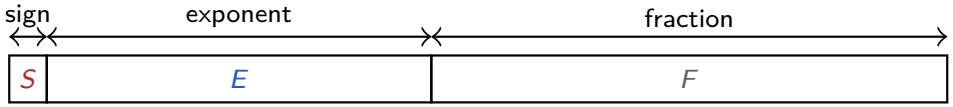

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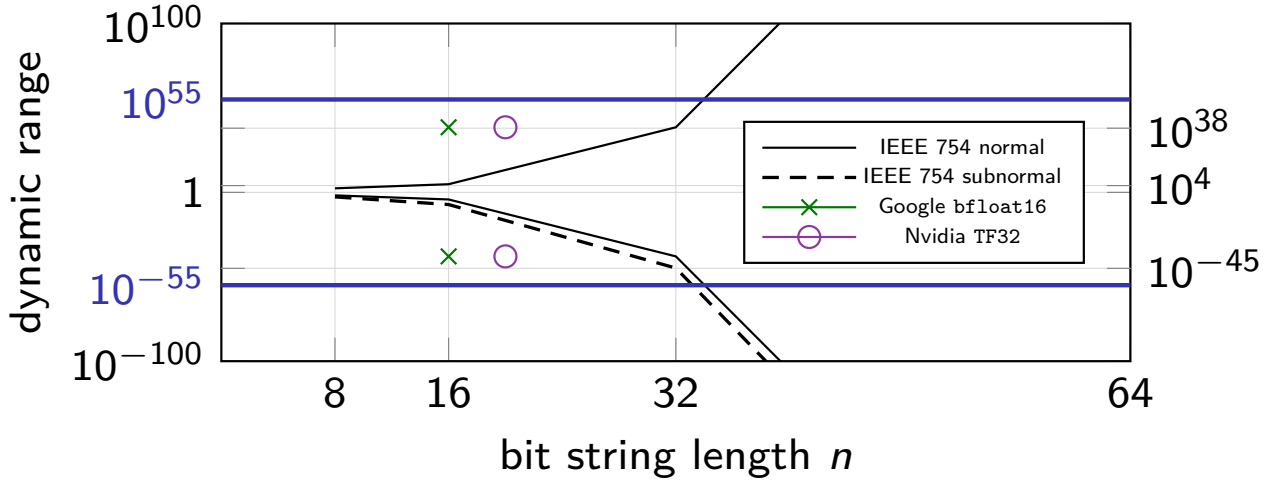

    Violated: too small for $n \leq 32$, too large for $n \geq 64$
  - Tapering

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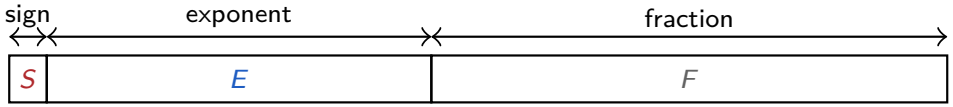

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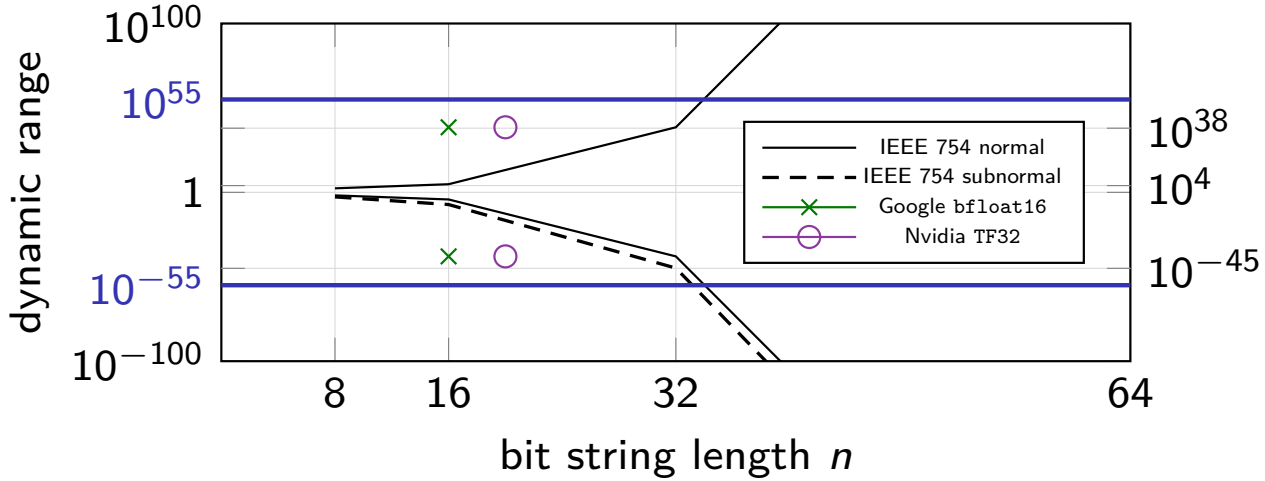

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  - Tapering: Numbers in magnitude close to one are used much more frequently than very large or very small numbers

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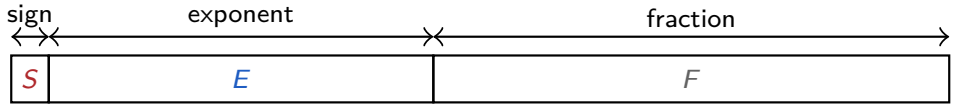

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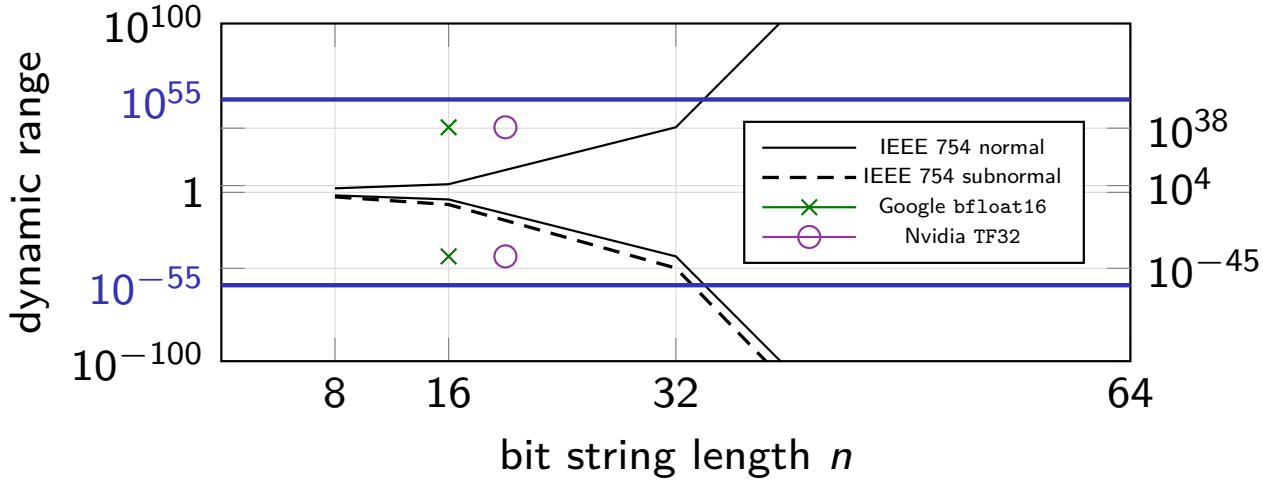

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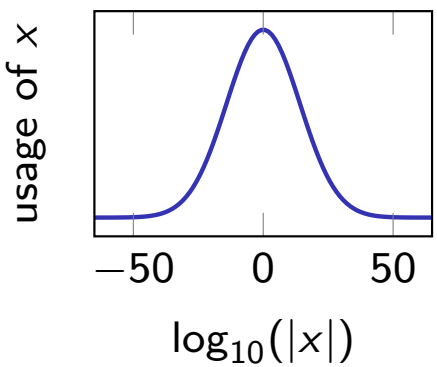

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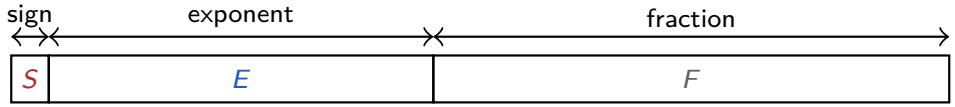

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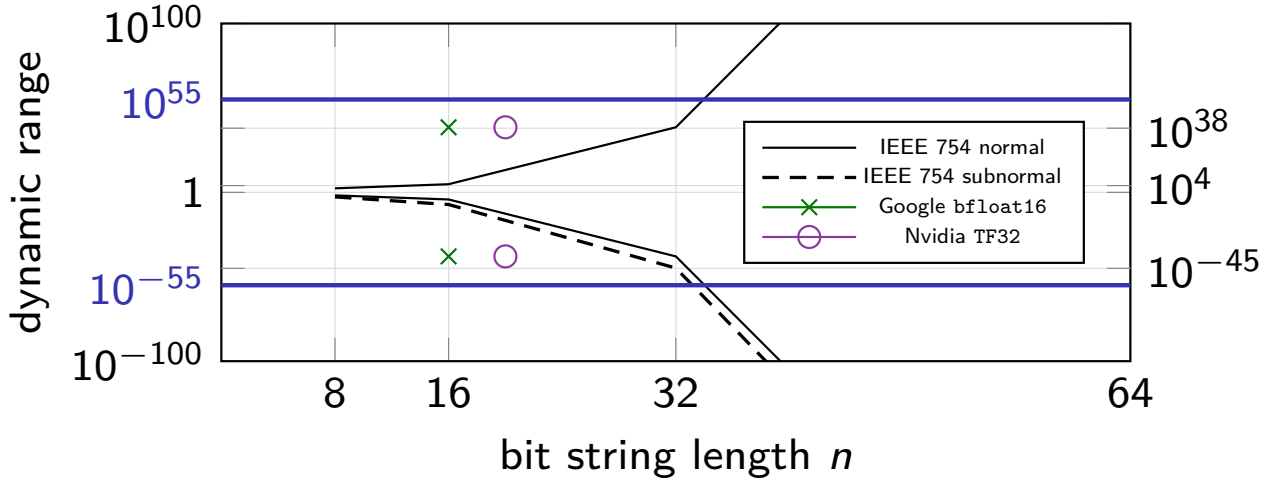

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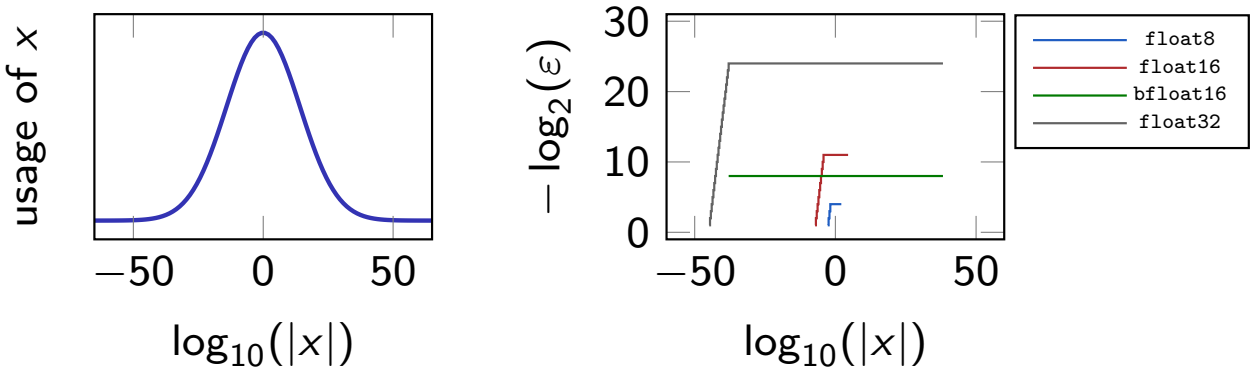

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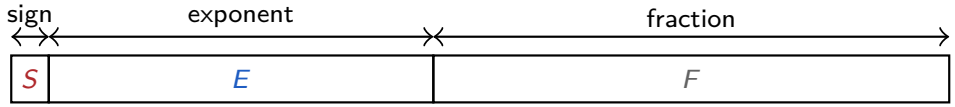

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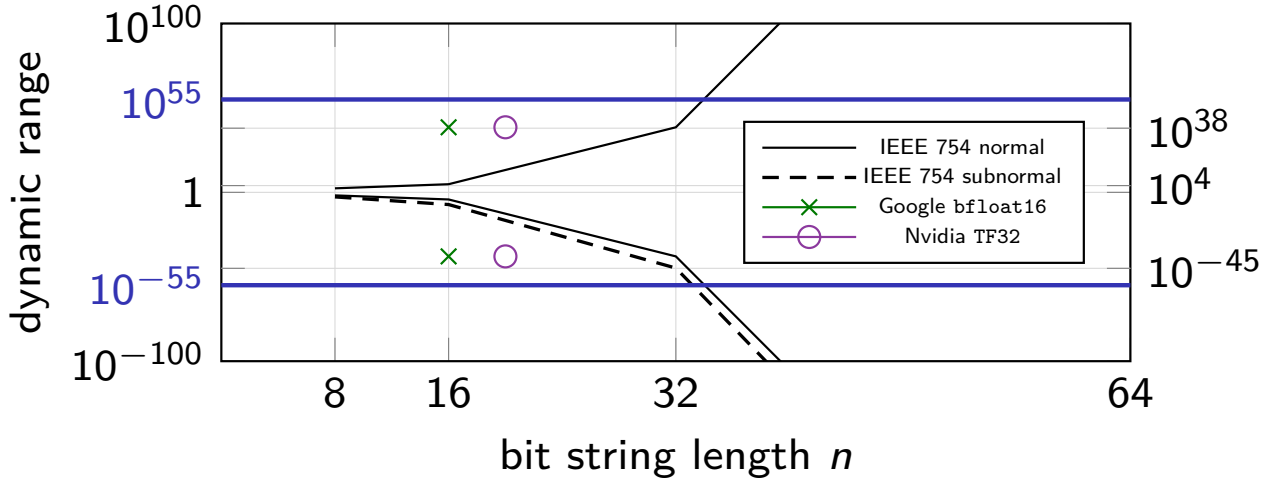

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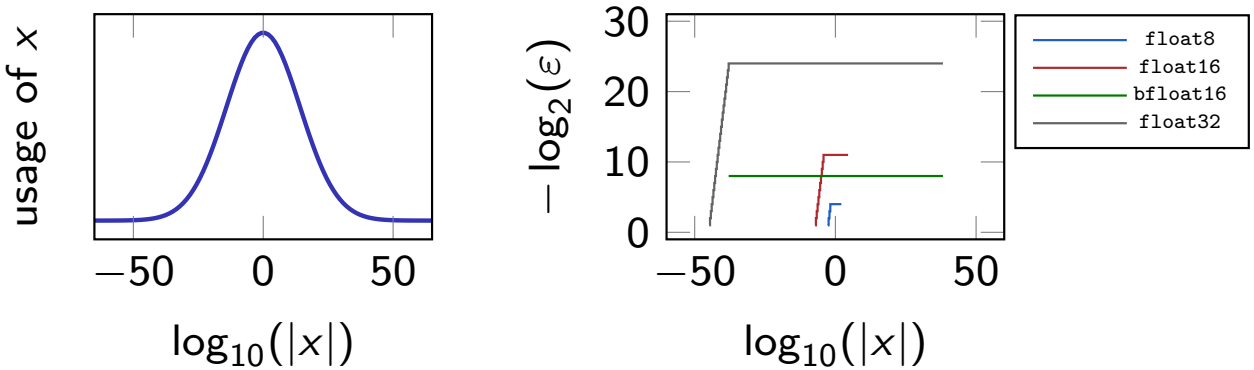

Violated

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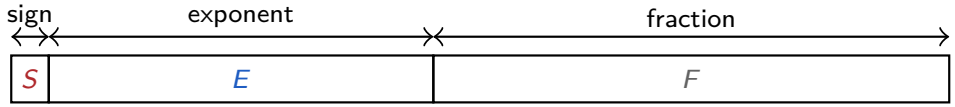

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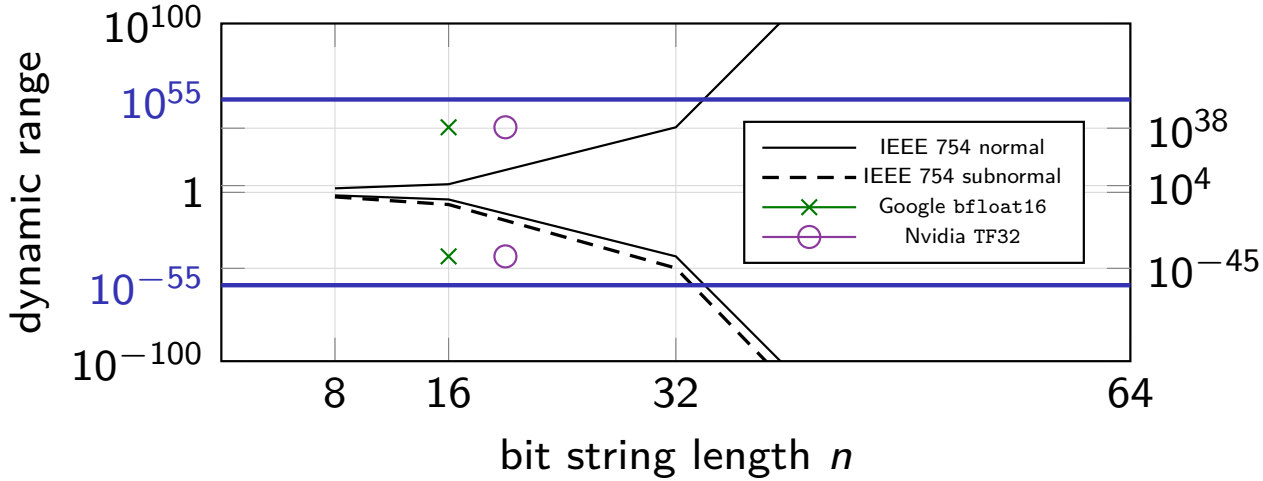

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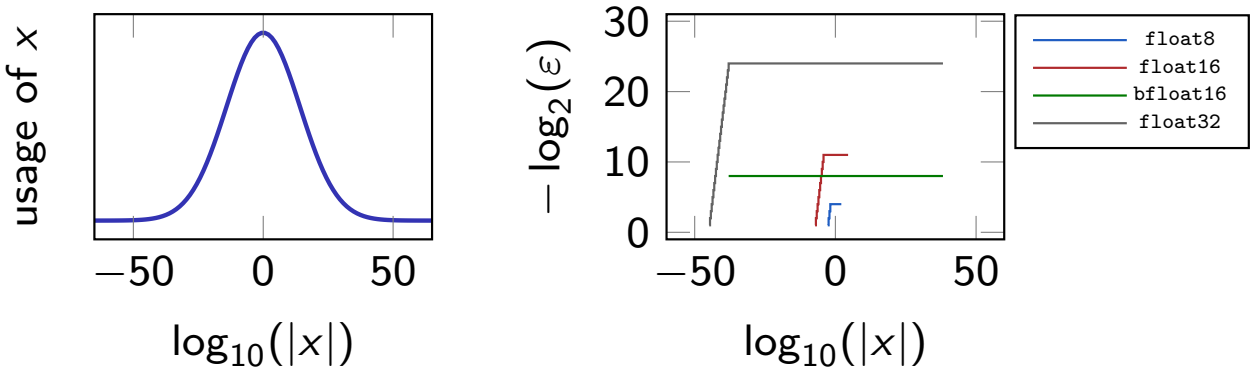

Violated: no tapering, uniform density (except subnormals)

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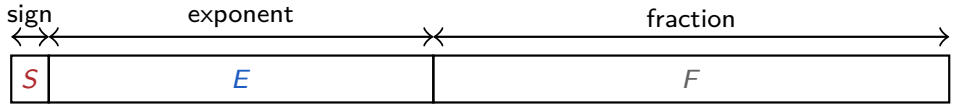
versus Desirable Design Criteria (2/2)



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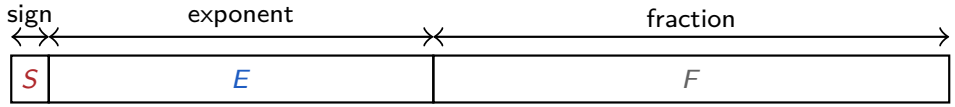
versus Desirable Design Criteria (2/2)



- No redundant encodings

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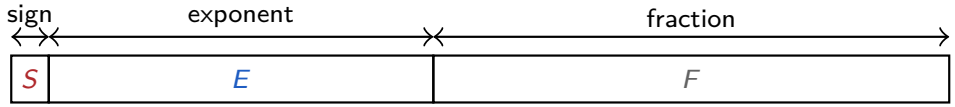
versus Desirable Design Criteria (2/2)



- No redundant encodings
  Violated

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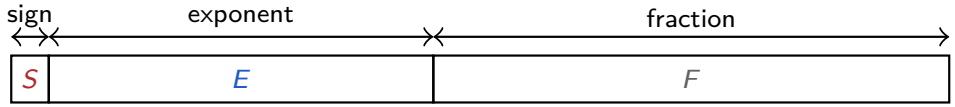
versus Desirable Design Criteria (2/2)



- No redundant encodings
  Violated: signed zero

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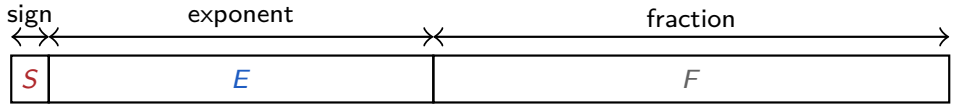
versus Desirable Design Criteria (2/2)



- No redundant encodings
  Violated: signed zero, redundant NaNs

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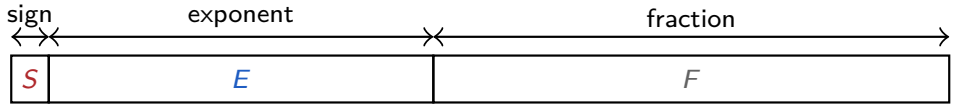
versus Desirable Design Criteria (2/2)



- No redundant encodings
  Violated: signed zero, redundant NaNs, excess numbers ($n \geq 64$)

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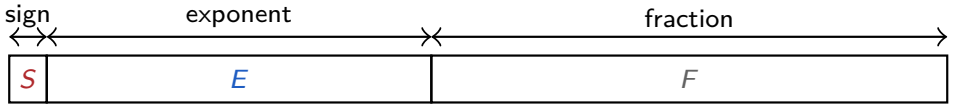
versus Desirable Design Criteria (2/2)



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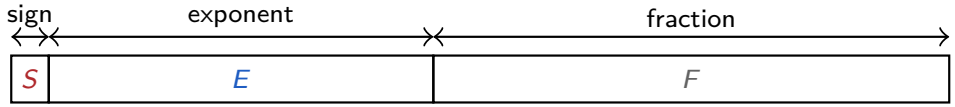
format	float8	float16	bfloat16	TF32	float32	float64	float128
waste/%	5.08	3.12	0.78	0.78	0.39	82.03	98.88

IEEE 754 Floating-Point

versus Desirable Design Criteria (2/2)



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- Defined for bit strings of any length

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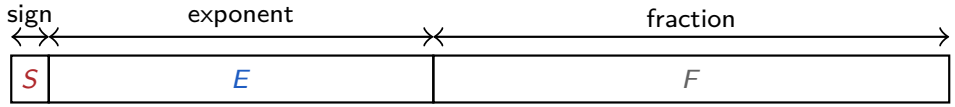
versus Desirable Design Criteria (2/2)



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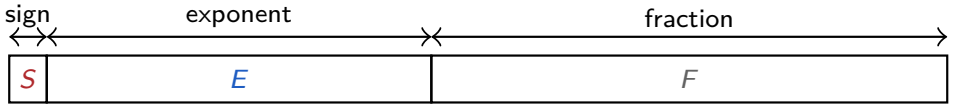

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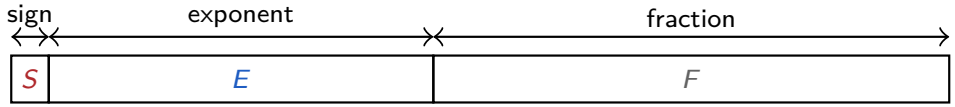
- Defined for bit strings of any length
  Violated: Only defined for $n \in \{8, 16, 32, 64, \dots\}$

IEEE 754 Floating-Point

versus Desirable Design Criteria (2/2)



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- Defined for bit strings of any length
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- Straightforward conversion between lengths

IEEE 754 Floating-Point

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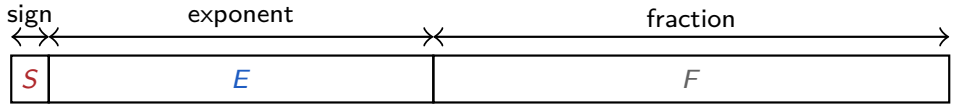

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- Straightforward conversion between lengths
  Violated

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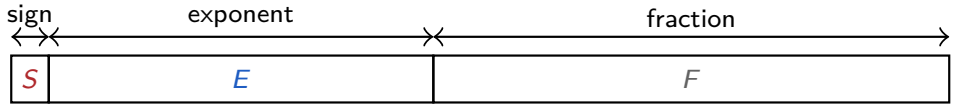

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  Violated: Requires exponent-reencoding

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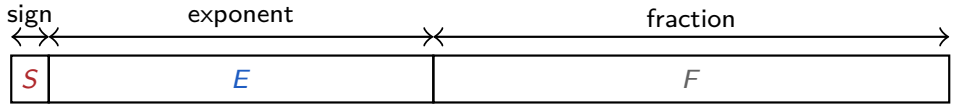

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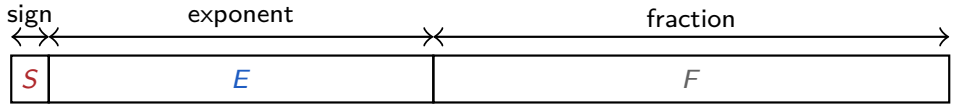
- Defined for bit strings of any length
  Violated: Only defined for $n \in \{8, 16, 32, 64, \ldots\}$
- Straightforward conversion between lengths
  Violated: Requires exponent-reencoding
- Simple/efficient hardware implementation (transistor count, energy efficiency, latency, throughput, etc.)

IEEE 754 Floating-Point

versus Desirable Design Criteria (2/2)



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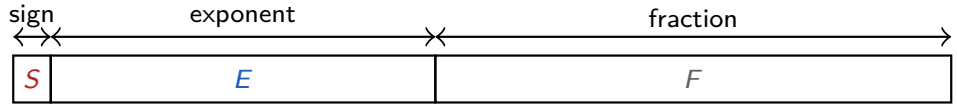

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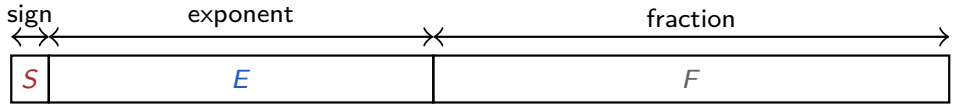

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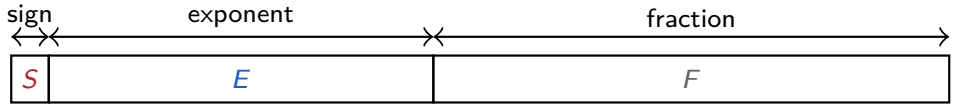

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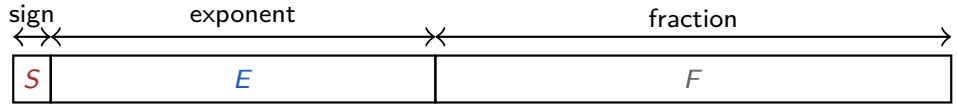

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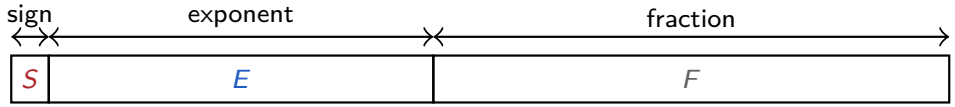

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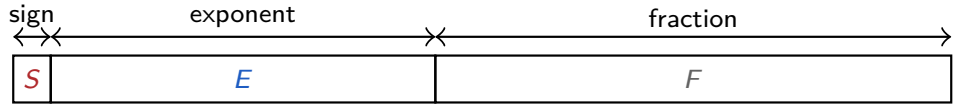

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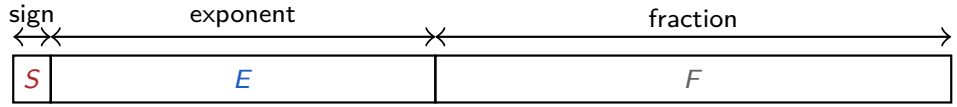

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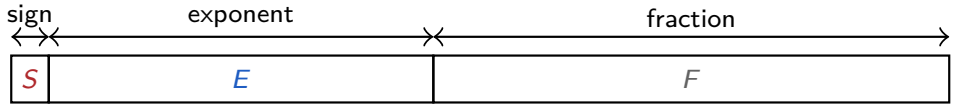

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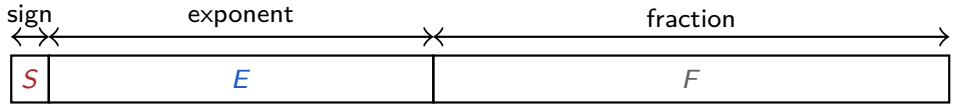

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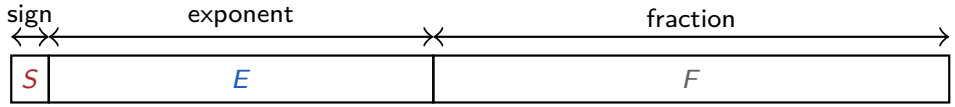

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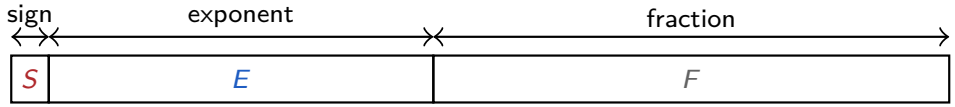

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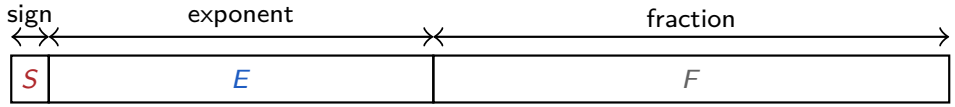

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IEEE 754 Floating-Point

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End of ‘double precision everywhere’ era

IEEE 754 Floating-Point

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- Time for a new paradigm?

Tapered Precision Machine Numbers

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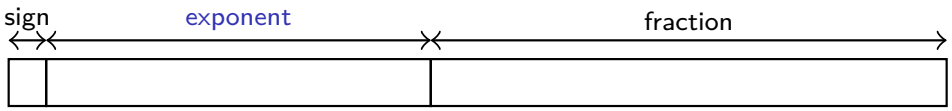
- Consider general floating-point format



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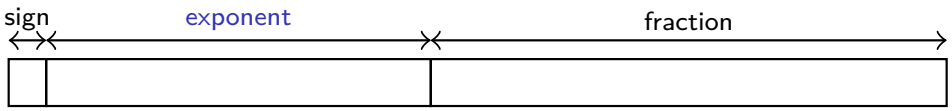

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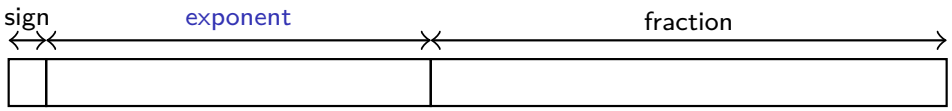

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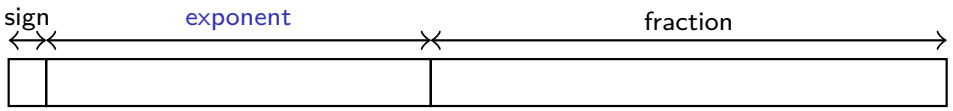

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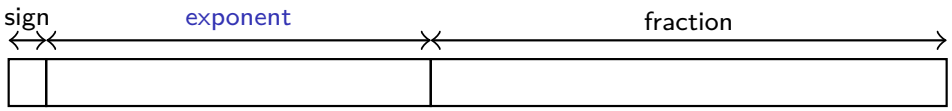

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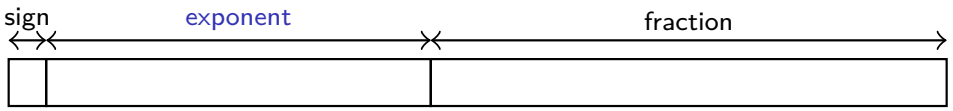

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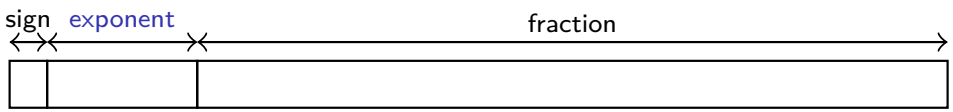

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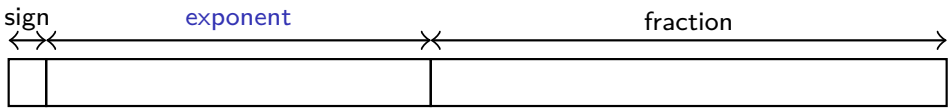

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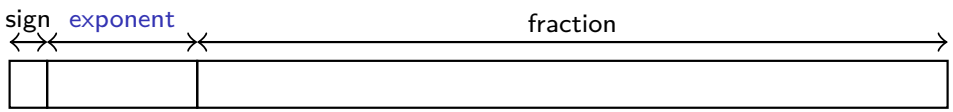

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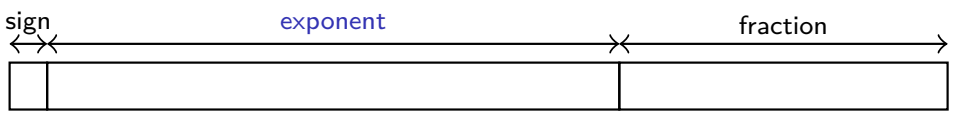

Posit Arithmetic

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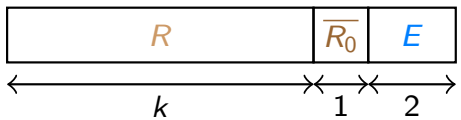

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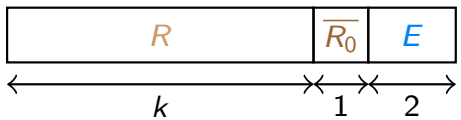

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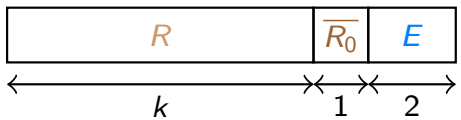

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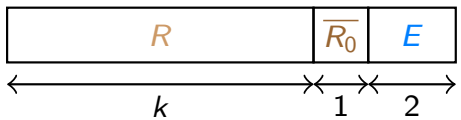

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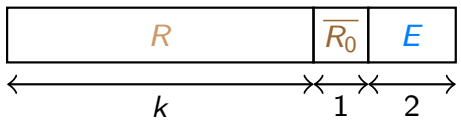

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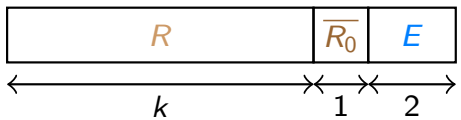

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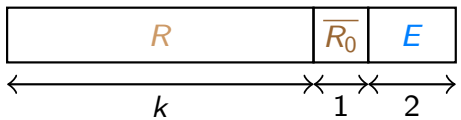

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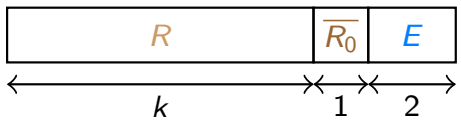

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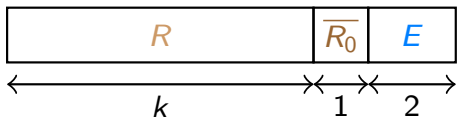

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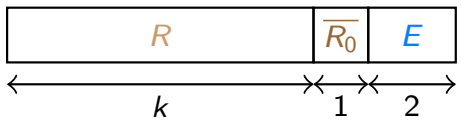

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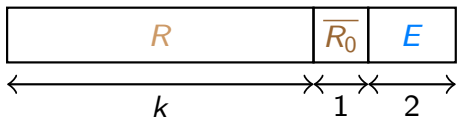

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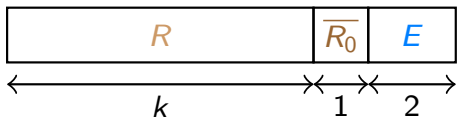

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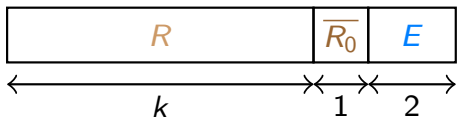

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  - Defined for all n, conversion between lengths simple rounding or expansion

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Posit Arithmetic

Discussion

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Tapering

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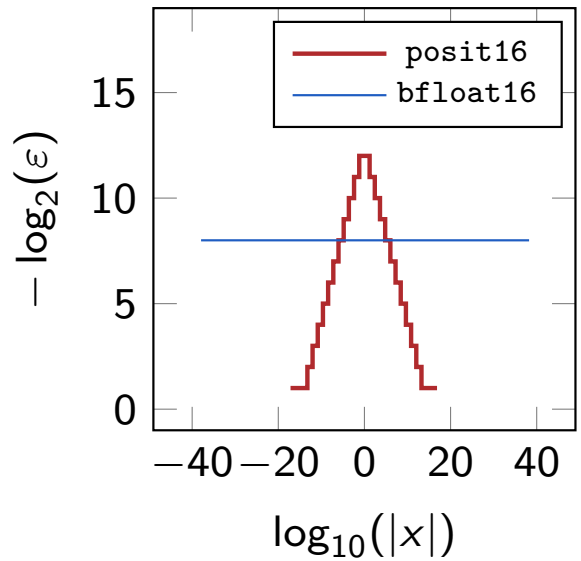

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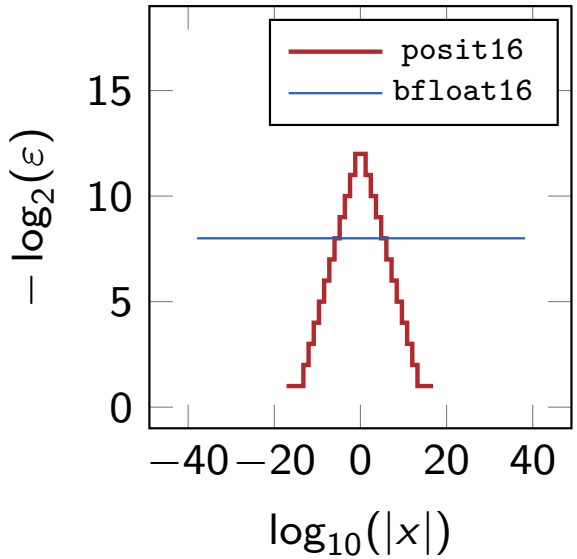

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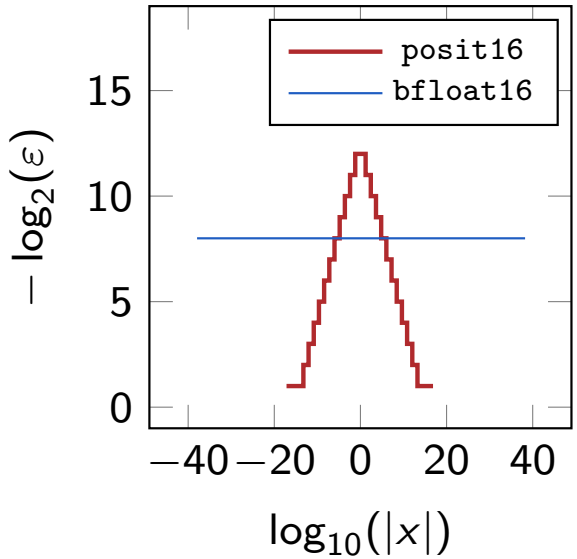

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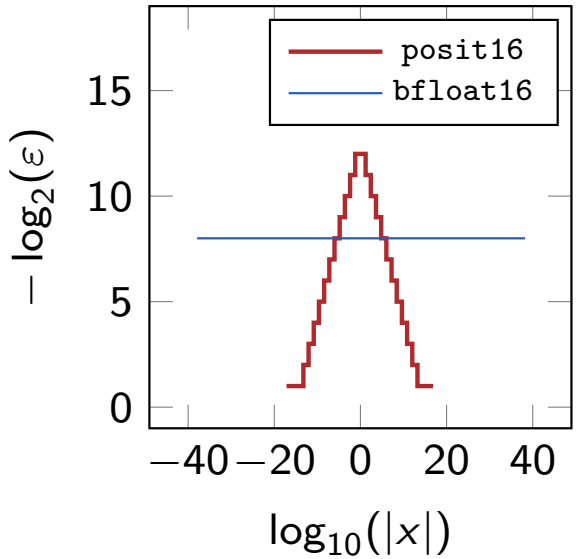

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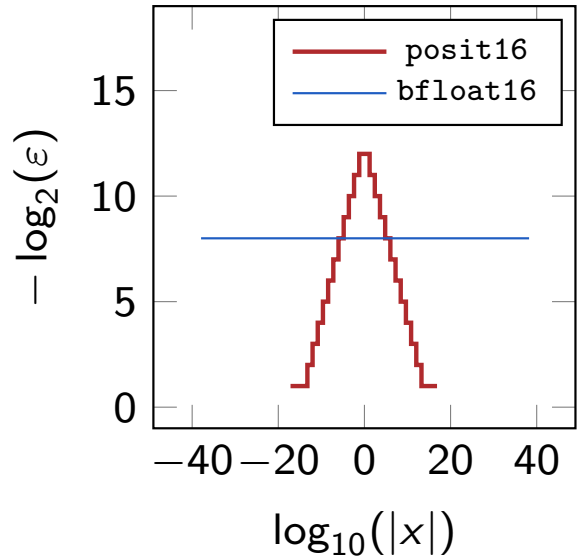

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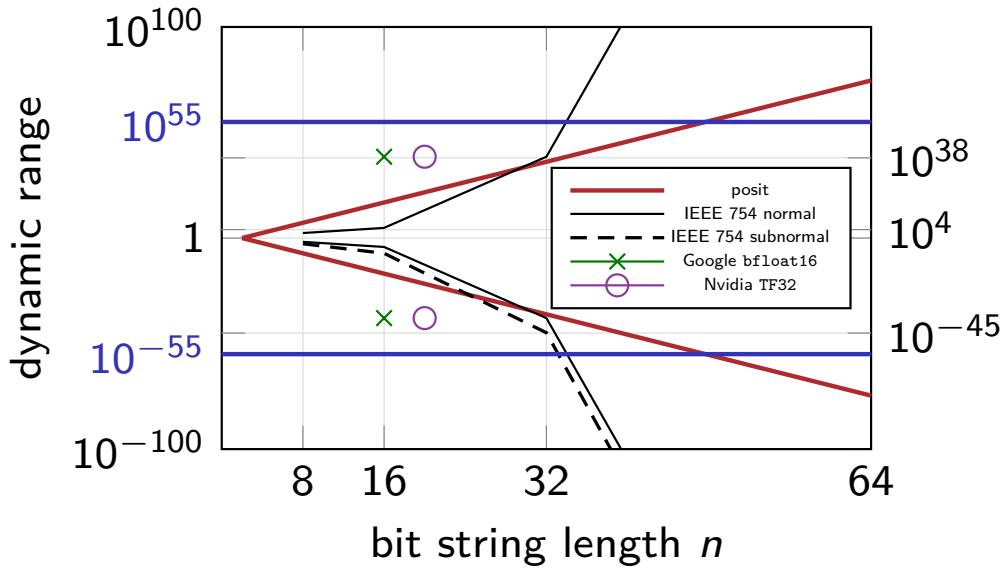

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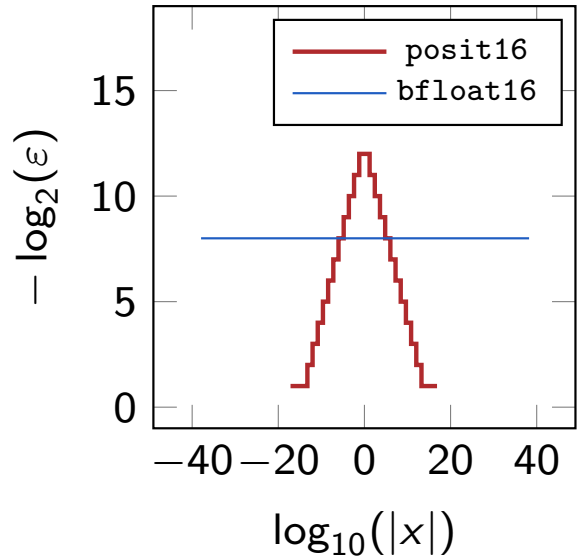

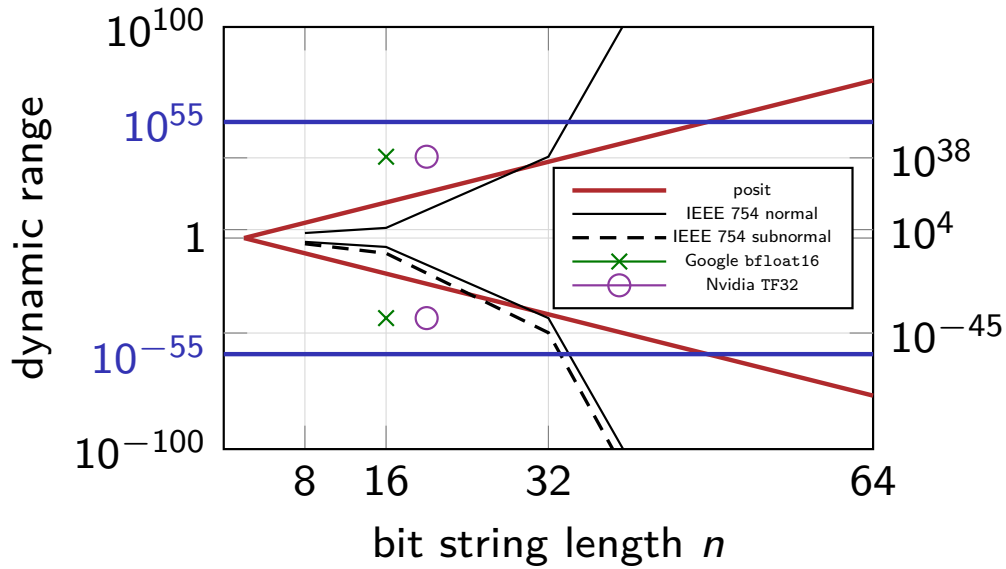

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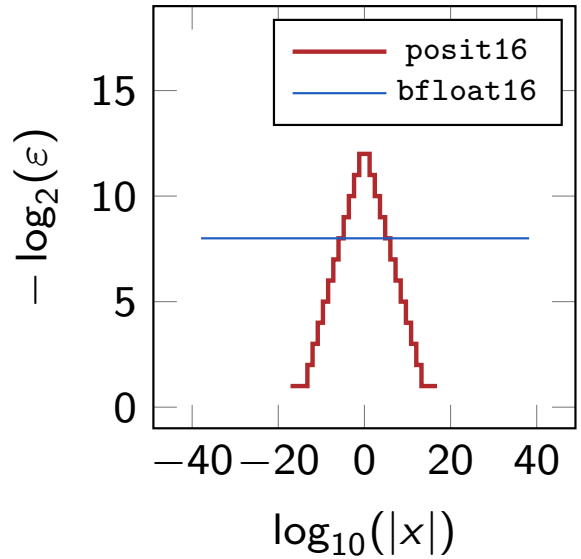

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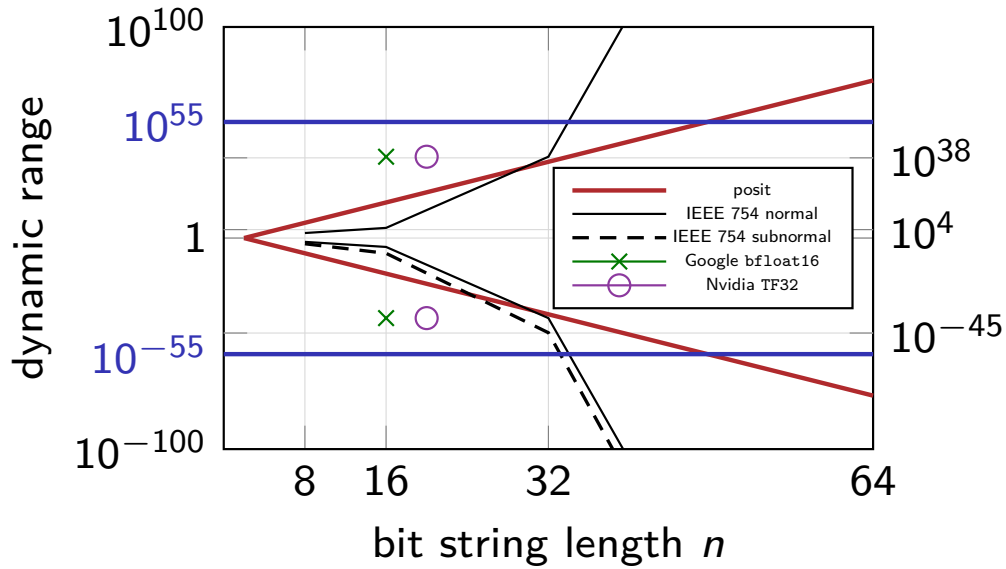

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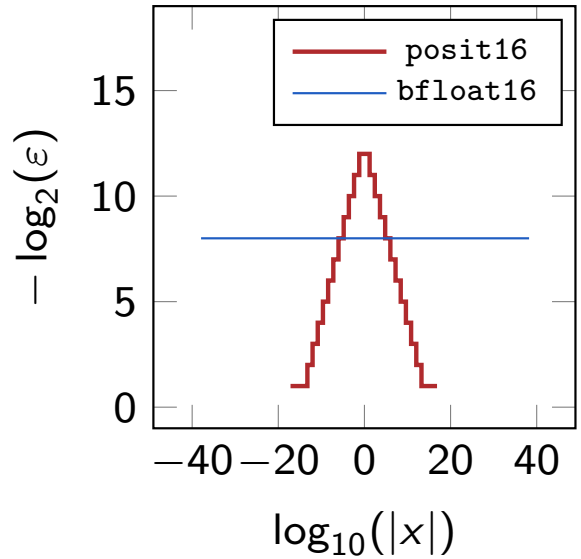

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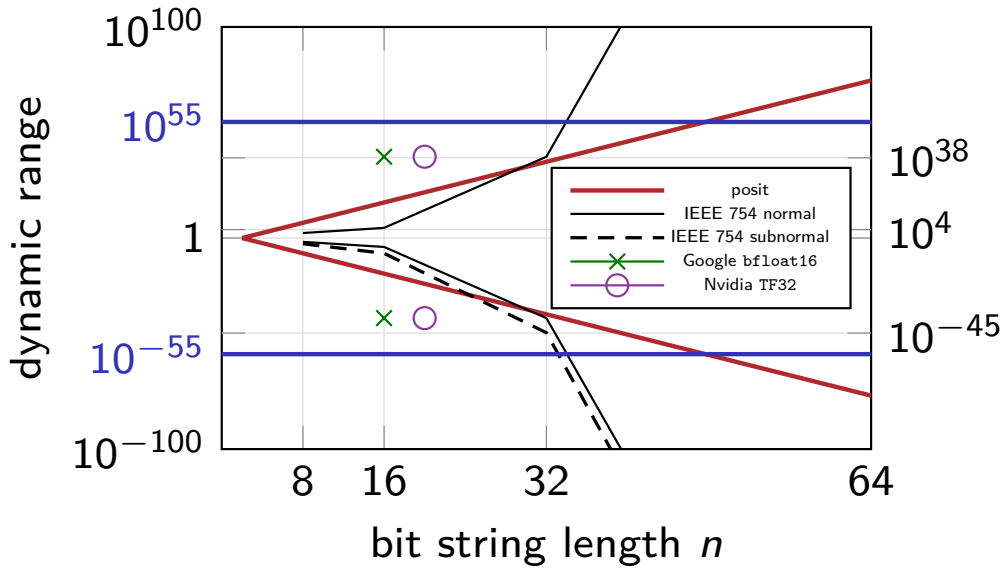

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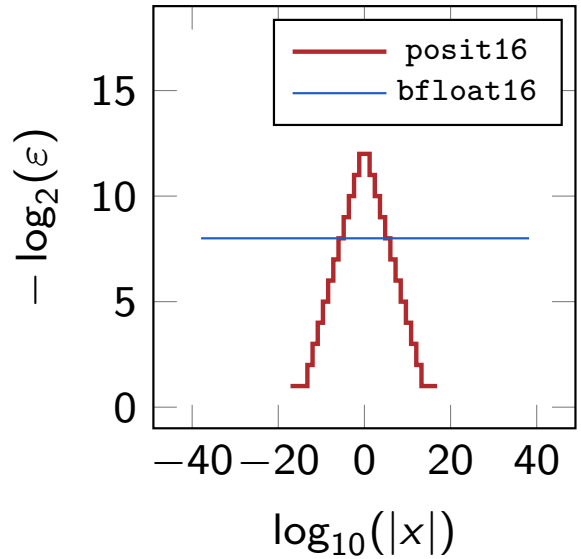

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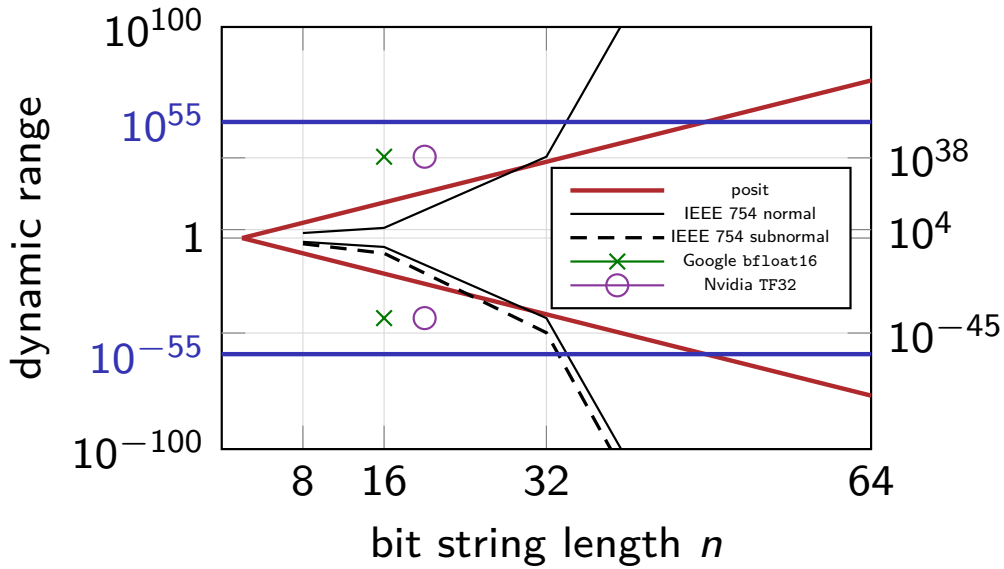

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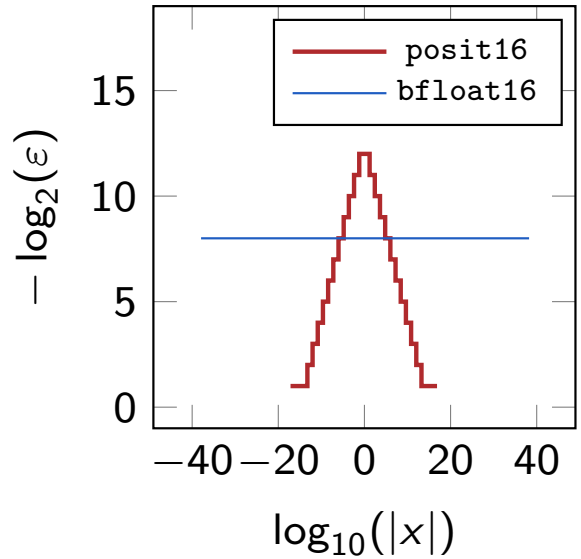

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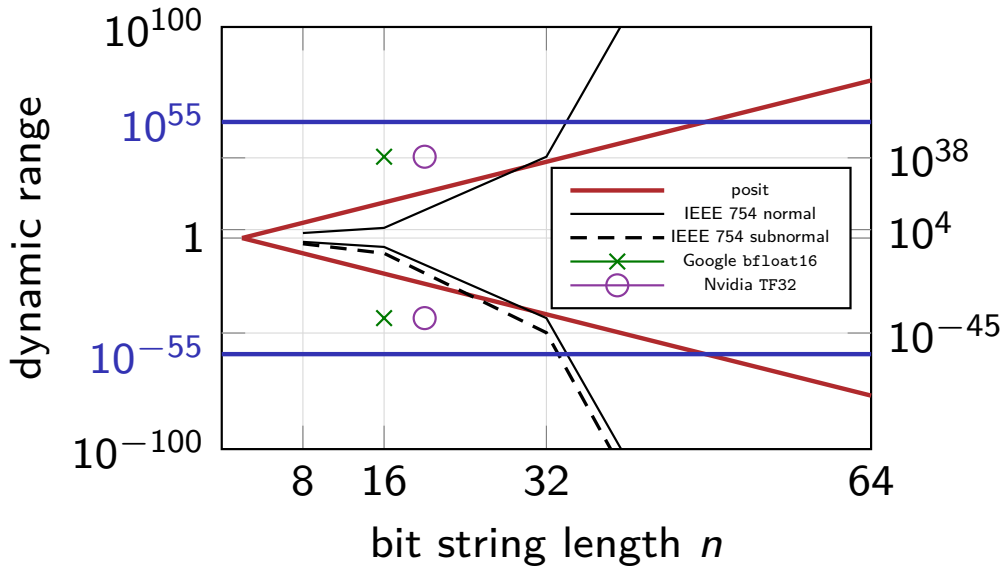

Takum Arithmetic

Overview and Exponent Coding

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  - Tapered format: Maximum exponent length must also be saturated

Takum Arithmetic

Overview and Exponent Coding

- Goals
  - More efficient exponent code
  - Logarithmic instead of linear tapering
  - Preserve useful posit properties
- Approach
  - Candidate sequence of 'saturated' integers $2^k - 1$ with $k \in \mathbb{N}_1$ $(1, 3, 7, 15, 31, 63, 127, 255, 511, 1023, \dots)$ for largest exponent value
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11111111111111111111111111111111111111111111111111111111111111010 (67b)

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Define takum as LNS with pure base $\sqrt{e}$ (transform to and from base e with ℓ-shift)

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Takum

Definition

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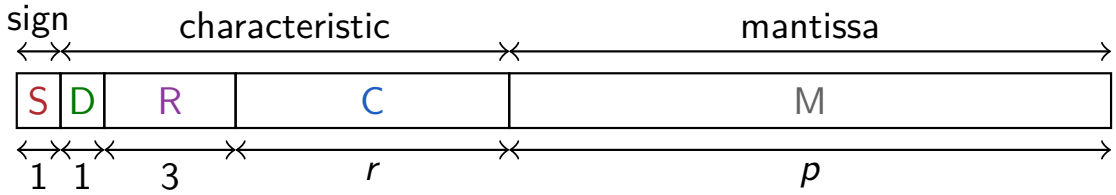

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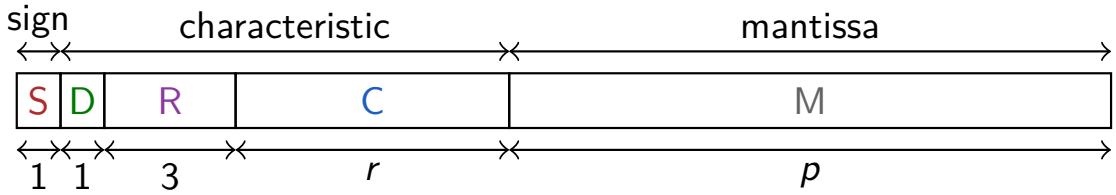

Sign bit S, direction bit D, regime bits R, characteristic bits C, mantissa bits M

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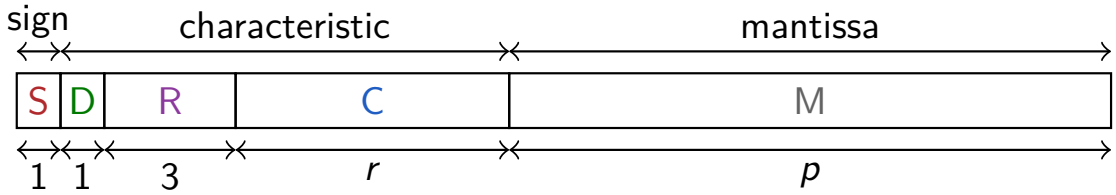

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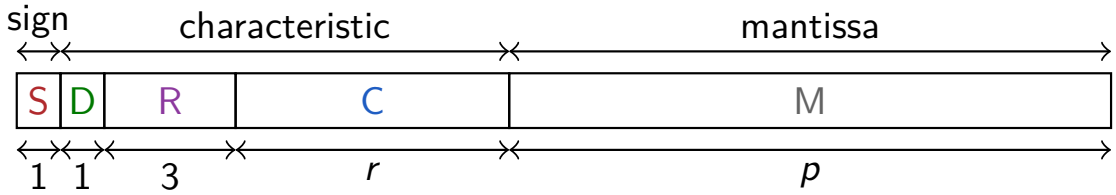
$$r := \begin{cases} \text{uint}\left(\overline{R}\right) & D = 0 \\ \text{uint}(R) & D = 1 \end{cases} \in \{0,\ldots,7\} \qquad \text{: regime}$$

Takum

Definition



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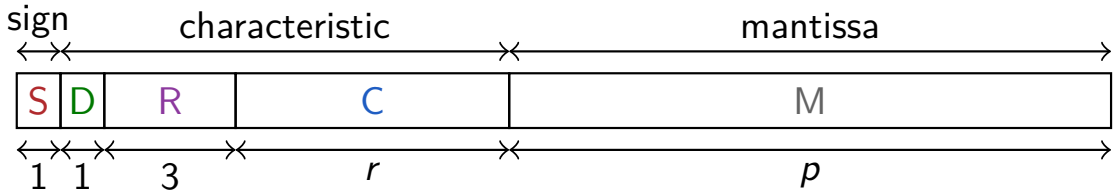

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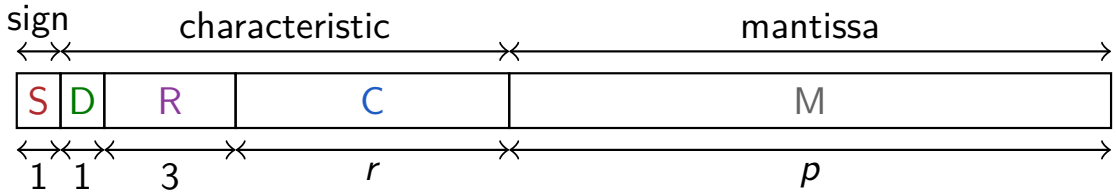

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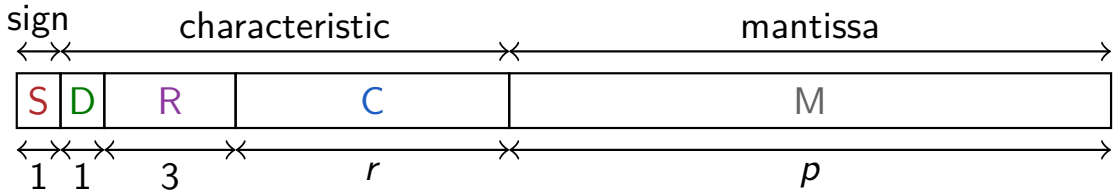

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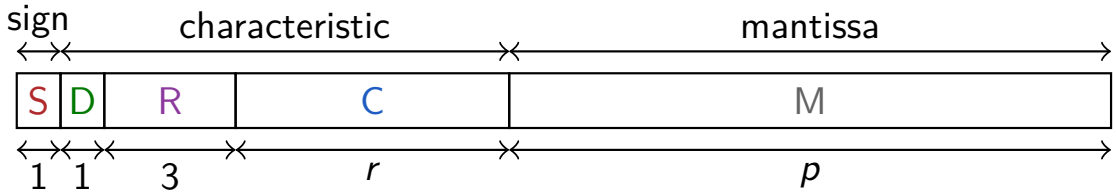

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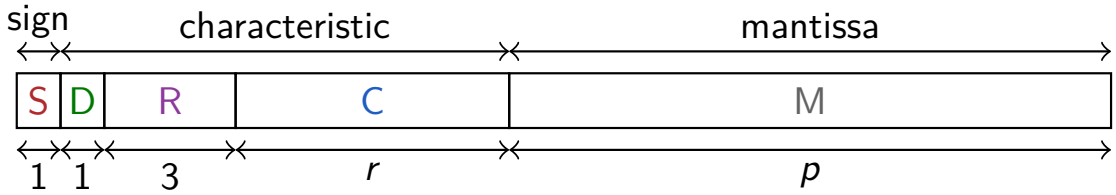

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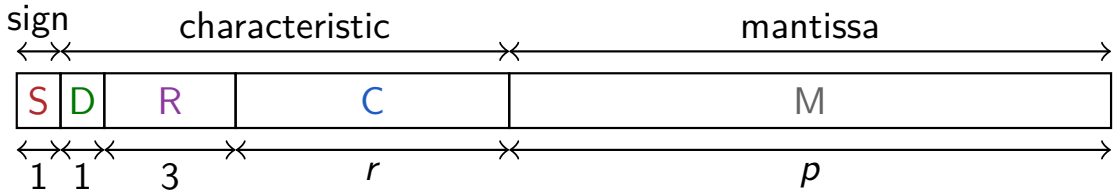

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Decoding

- Only 16 distinct cases via (D, R), just two LUTs (32 byte, 16 byte)

Takum

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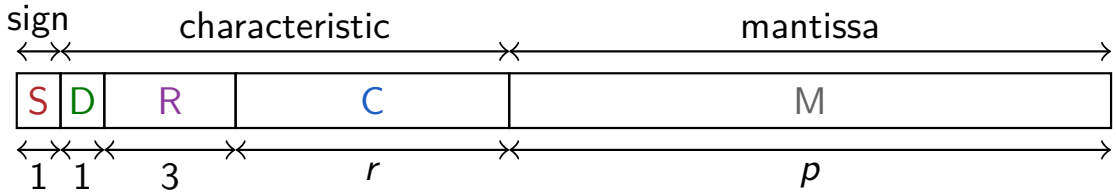

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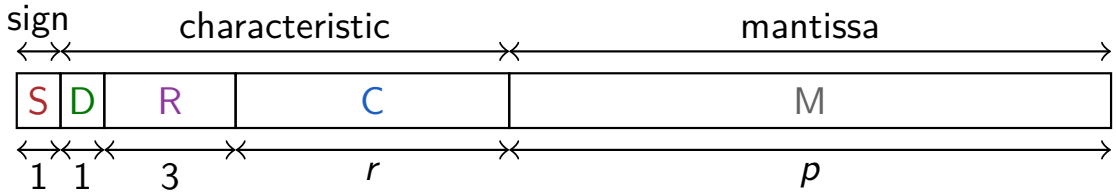

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- (D, R, C) is at most 11 bits long $\rightarrow$ shared logic for all n
- Computation of c is conditional negation, bitwise OR and increment

Takum

Dynamic Range

- Dynamic range $(\sqrt{\mathrm{e}}^{-255}, \sqrt{\mathrm{e}}^{255}) \approx (4.2 \times 10^{-56}, 2.4 \times 10^{55})$ for $n \geq 12$

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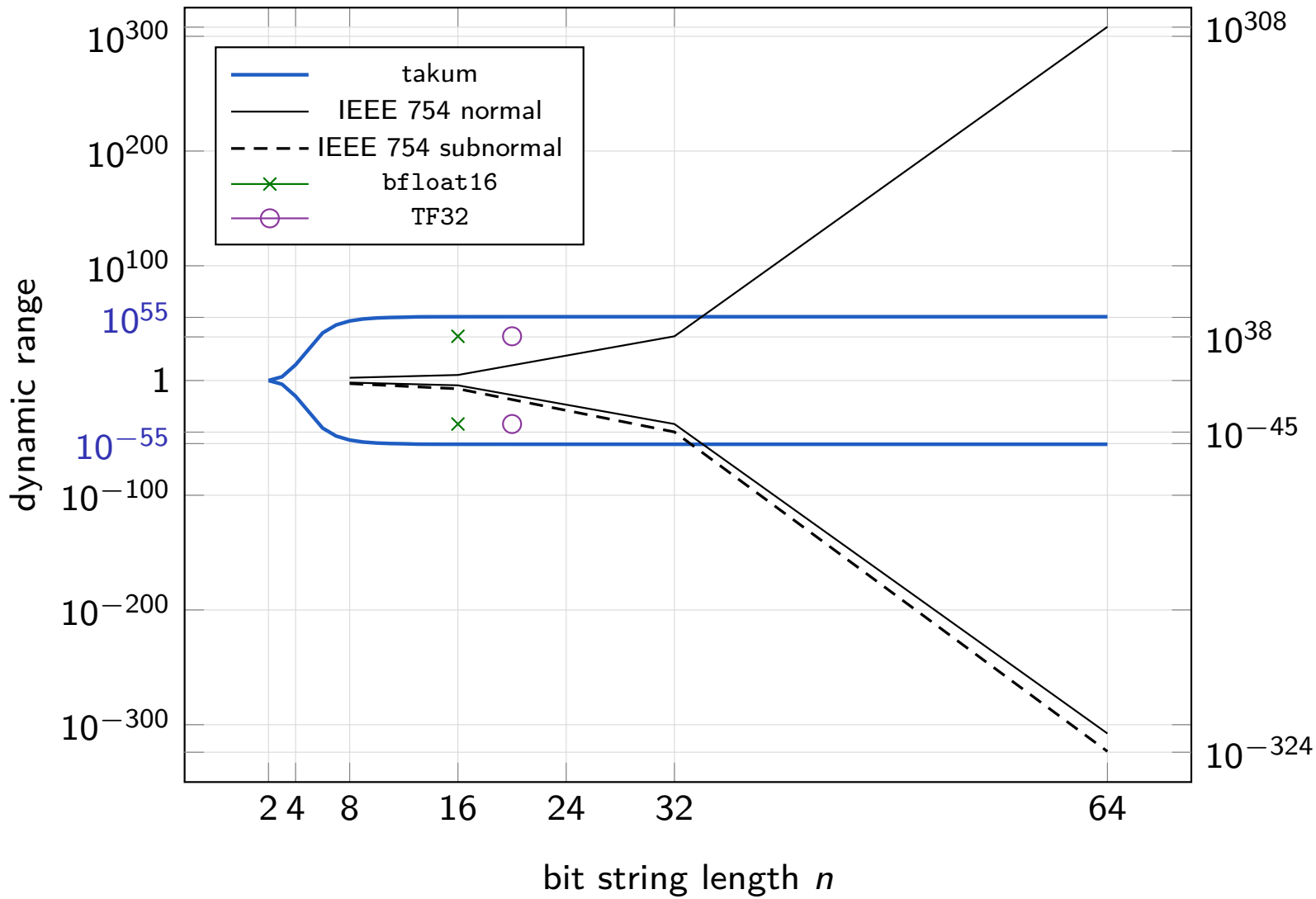
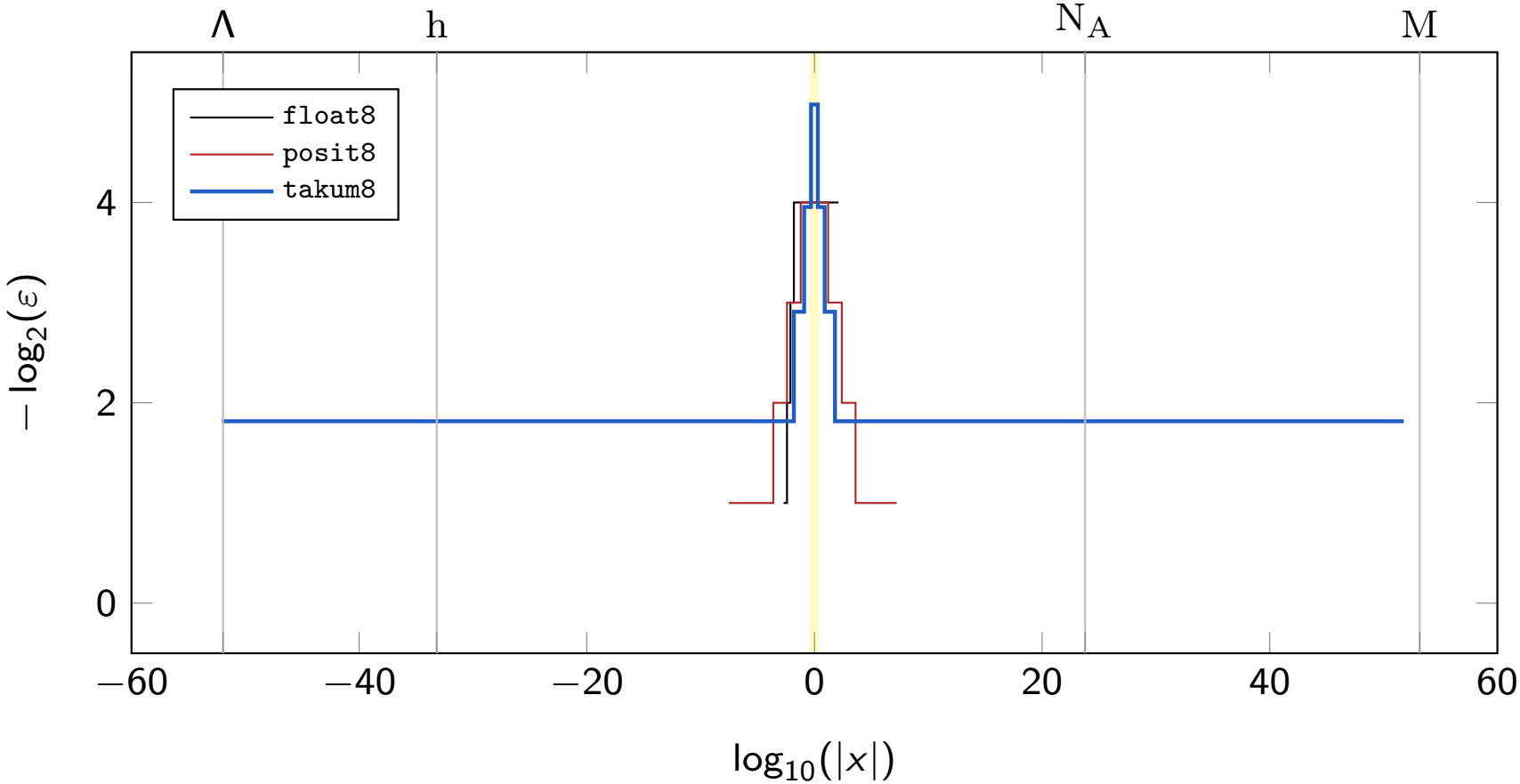


Takum

Density (8 Bit)



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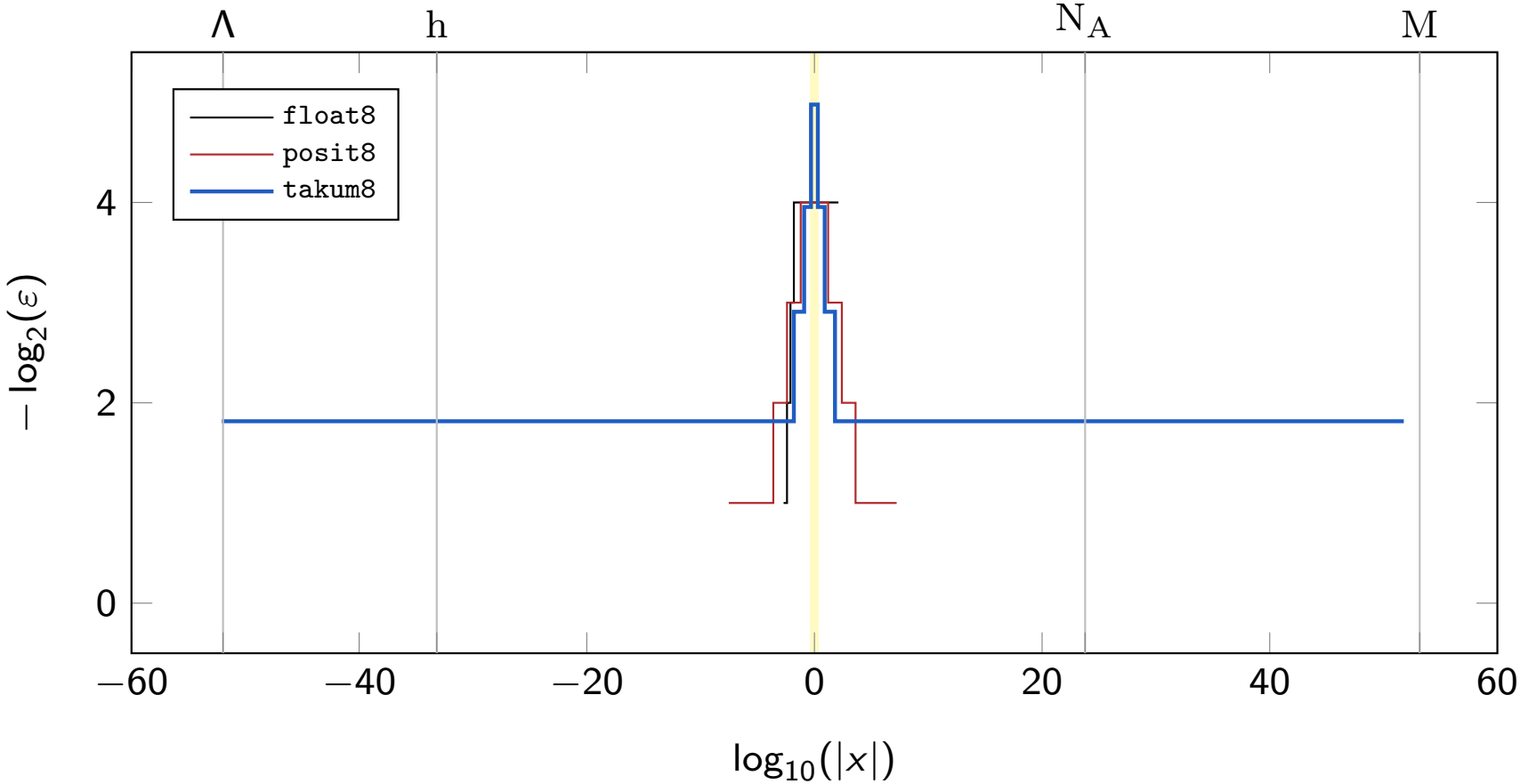

- `takum8` exceeds `float8` ('golden zone') and `posit8` near unity (2 times denser)

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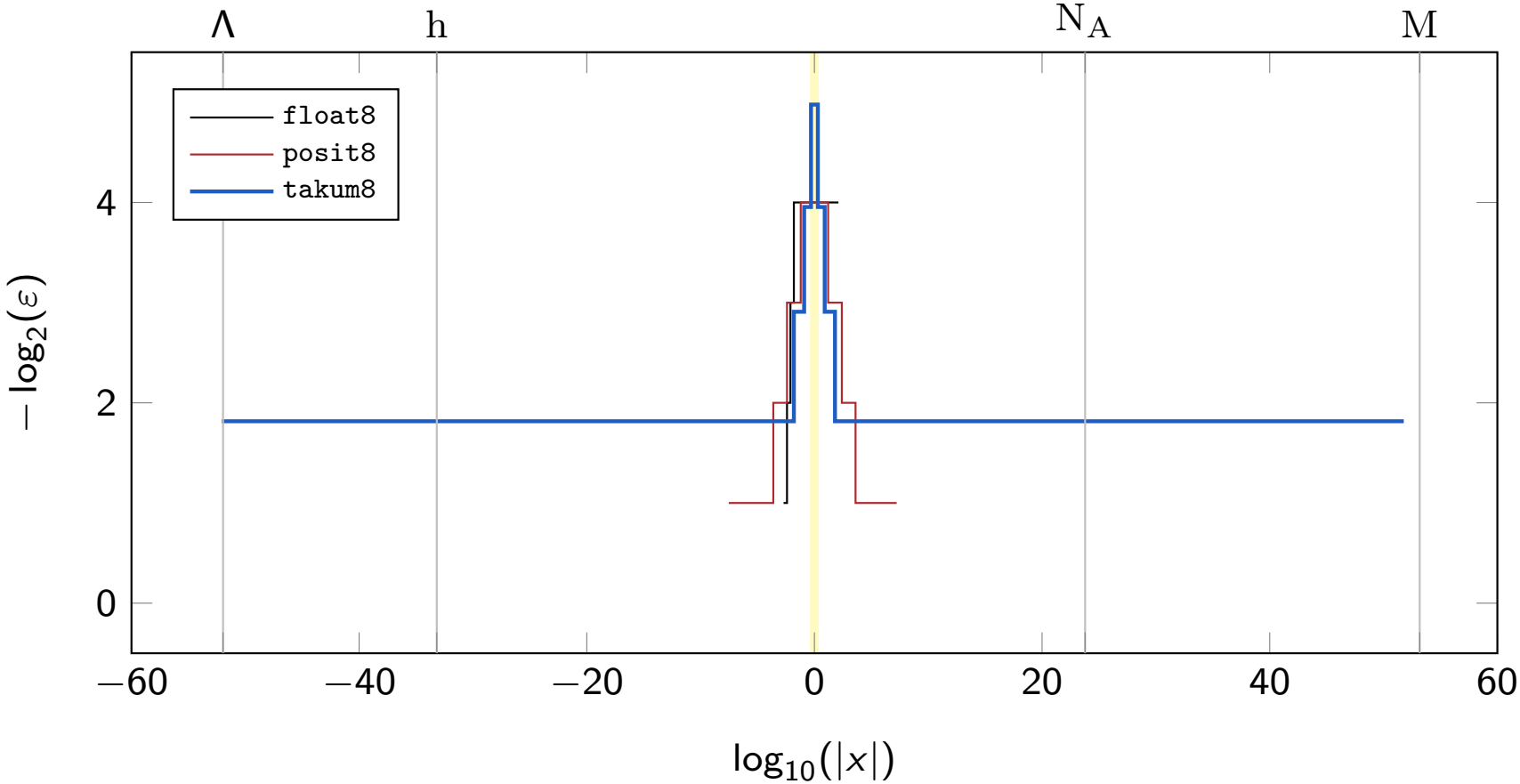
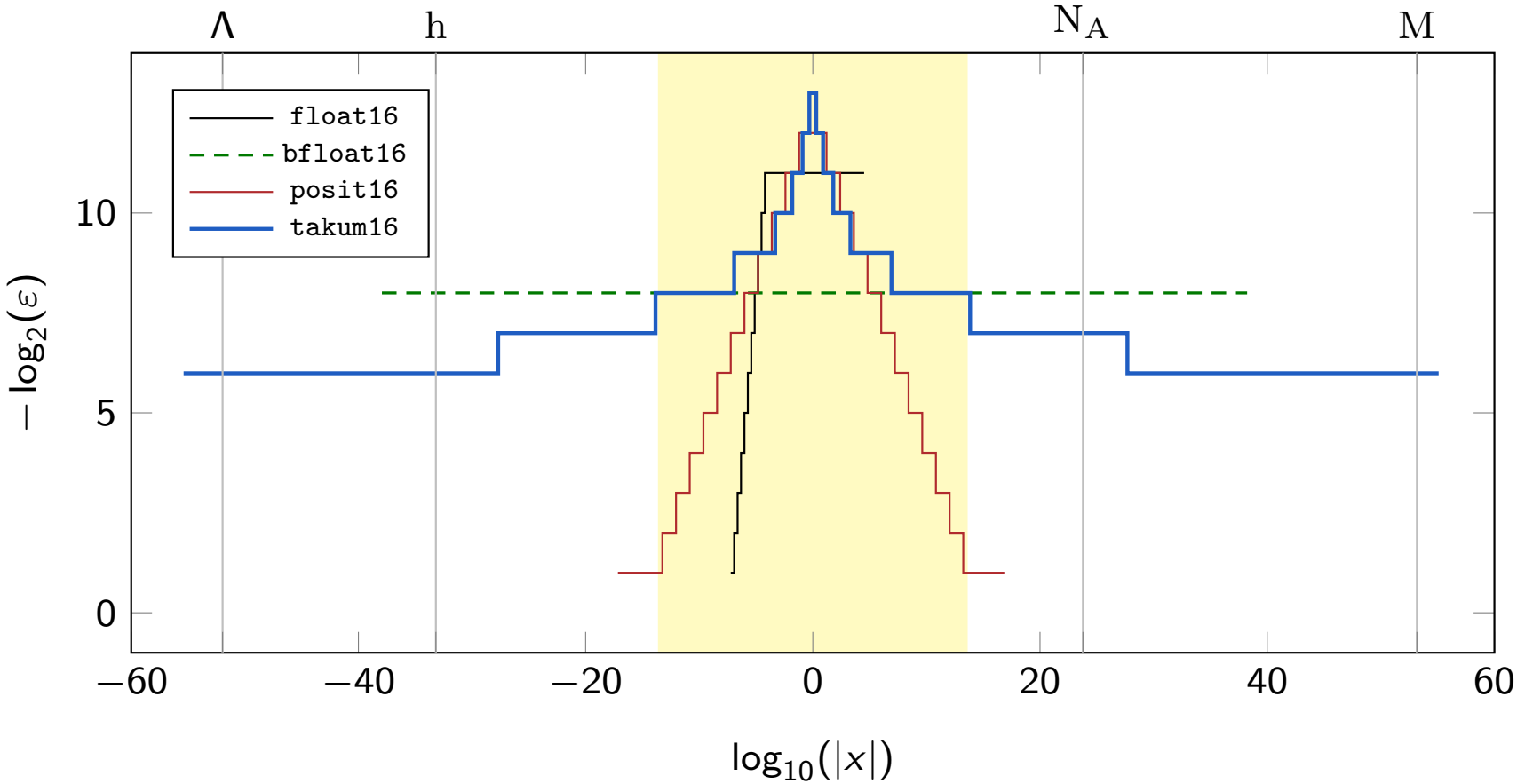


- `takum8` exceeds `float8` ('golden zone') and `posit8` near unity (2 times denser)
- Almost full dynamic range across benchmark constants Λ (cosmological constant), h (PLANCK constant), N_A (AVOGADRO constant), M (mass of the universe)

Takum

Density (16 Bit)



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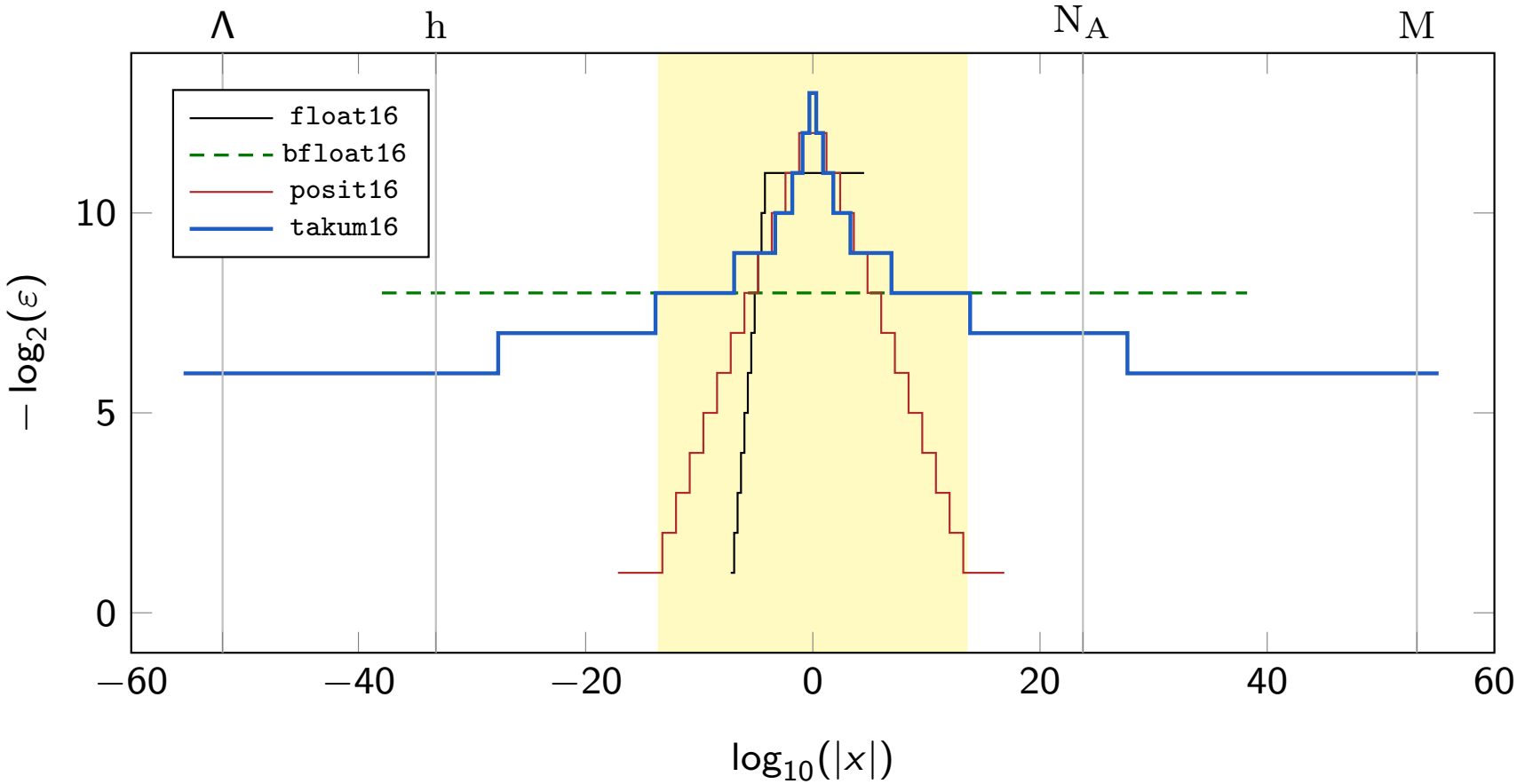

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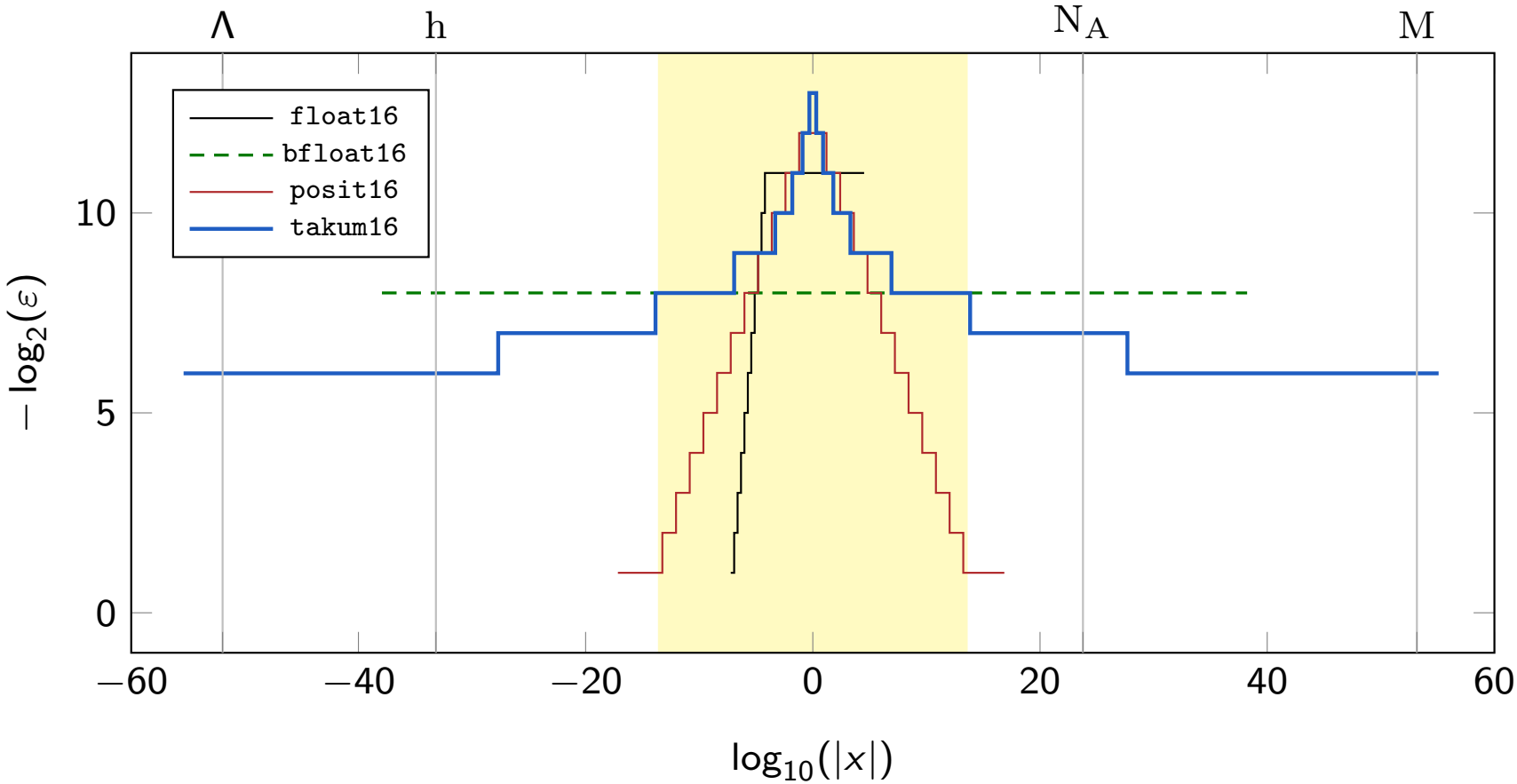

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Takum

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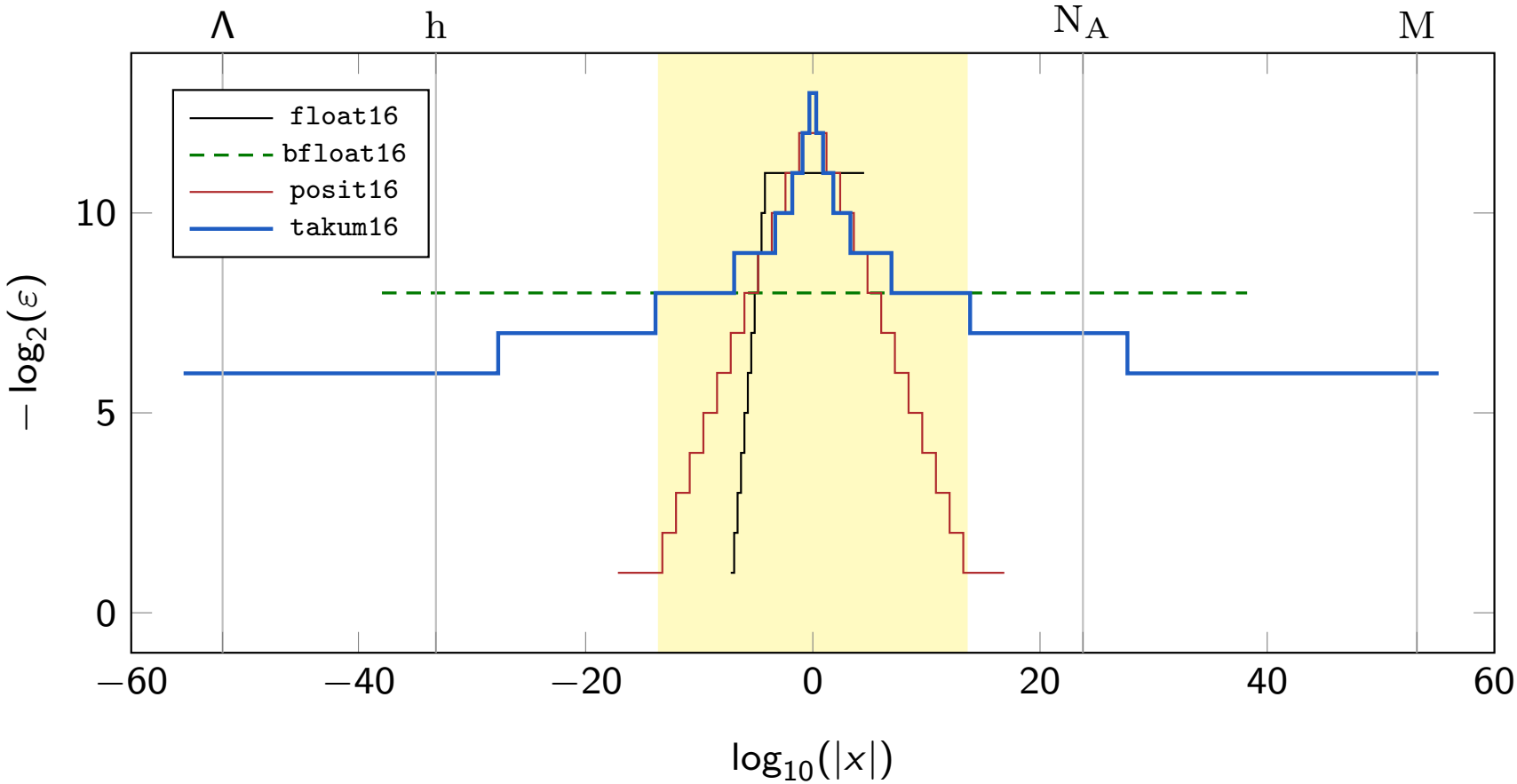

- Perfect logarithmic tapering
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Takum

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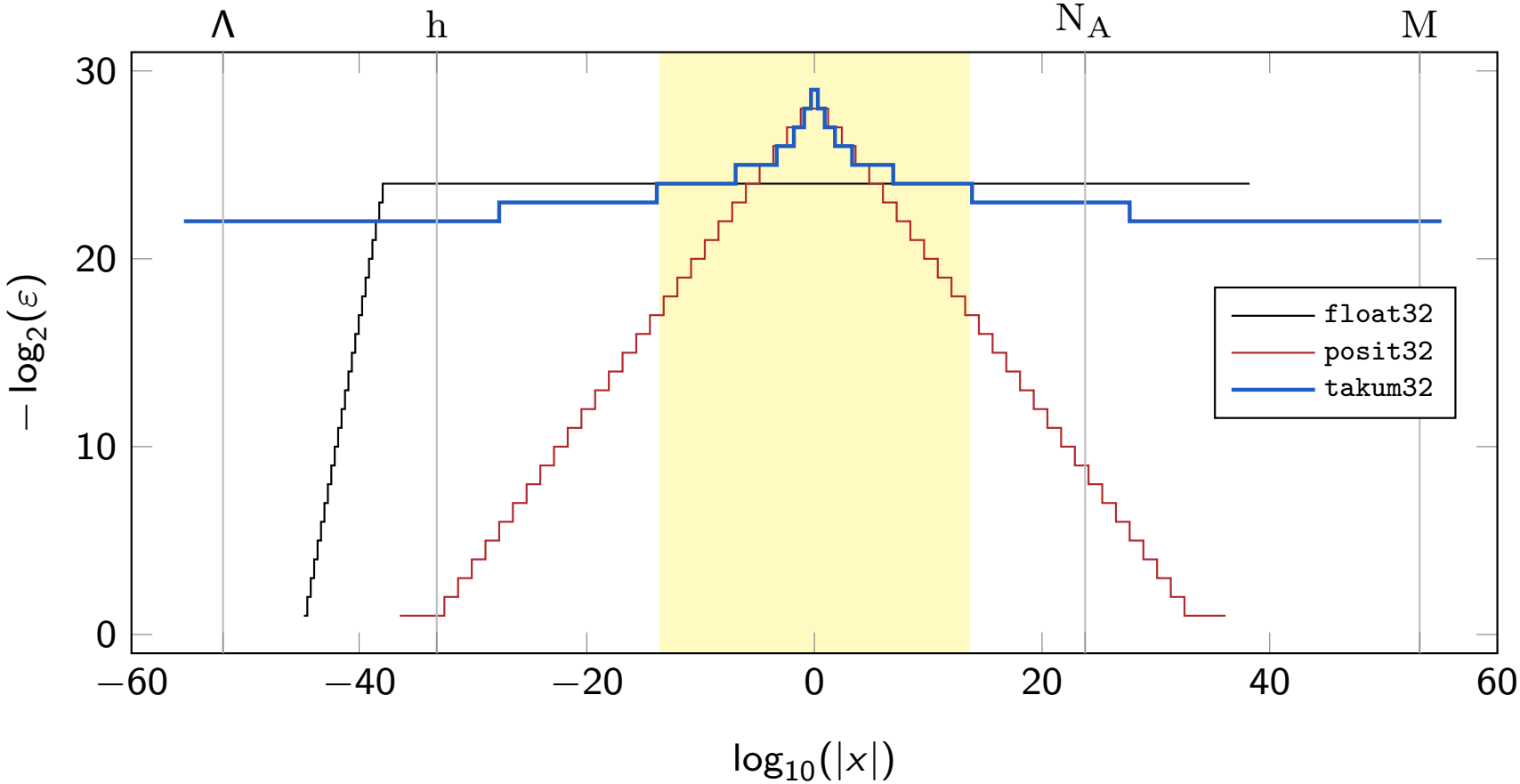

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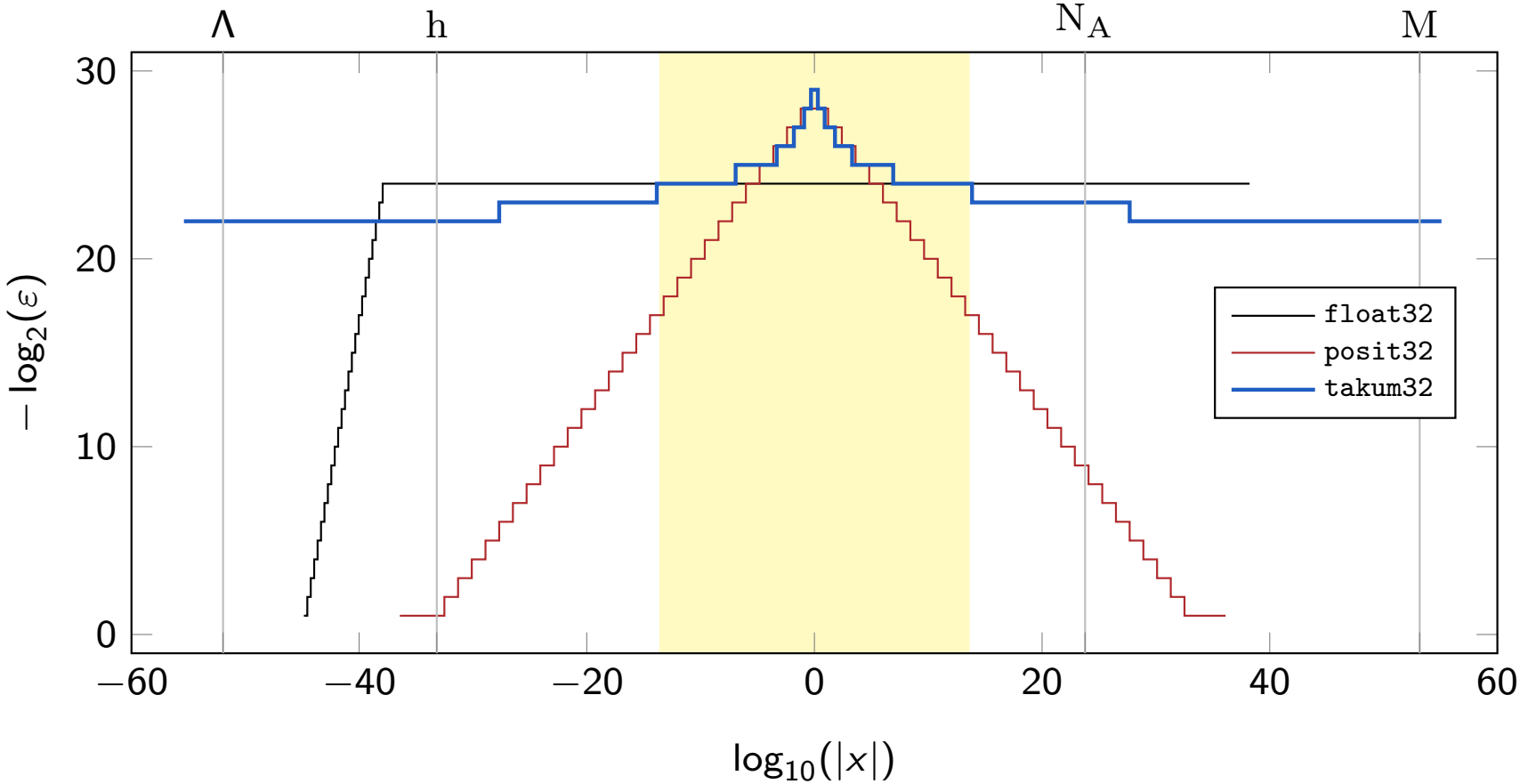

- Same golden zone as with $n = 16$

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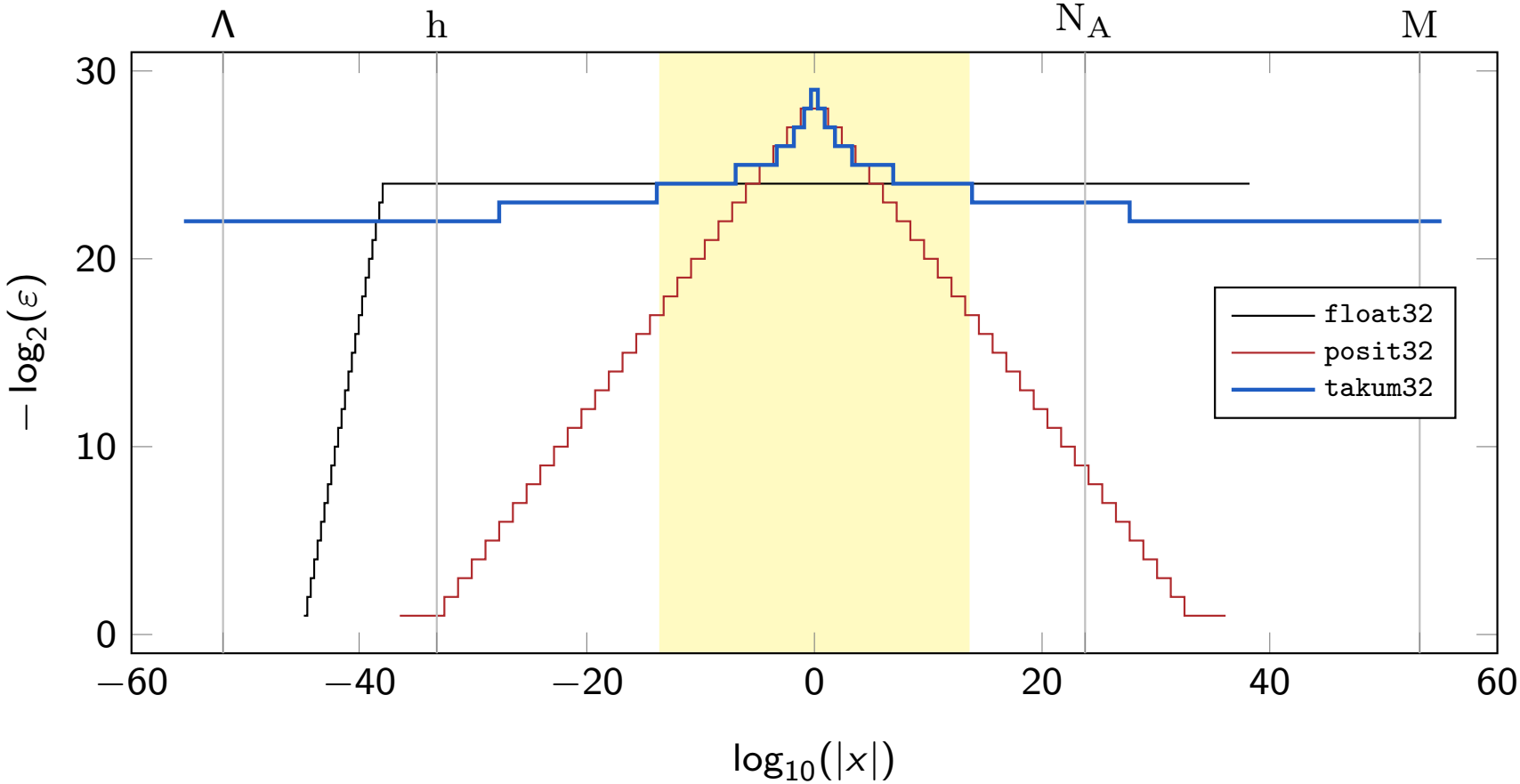
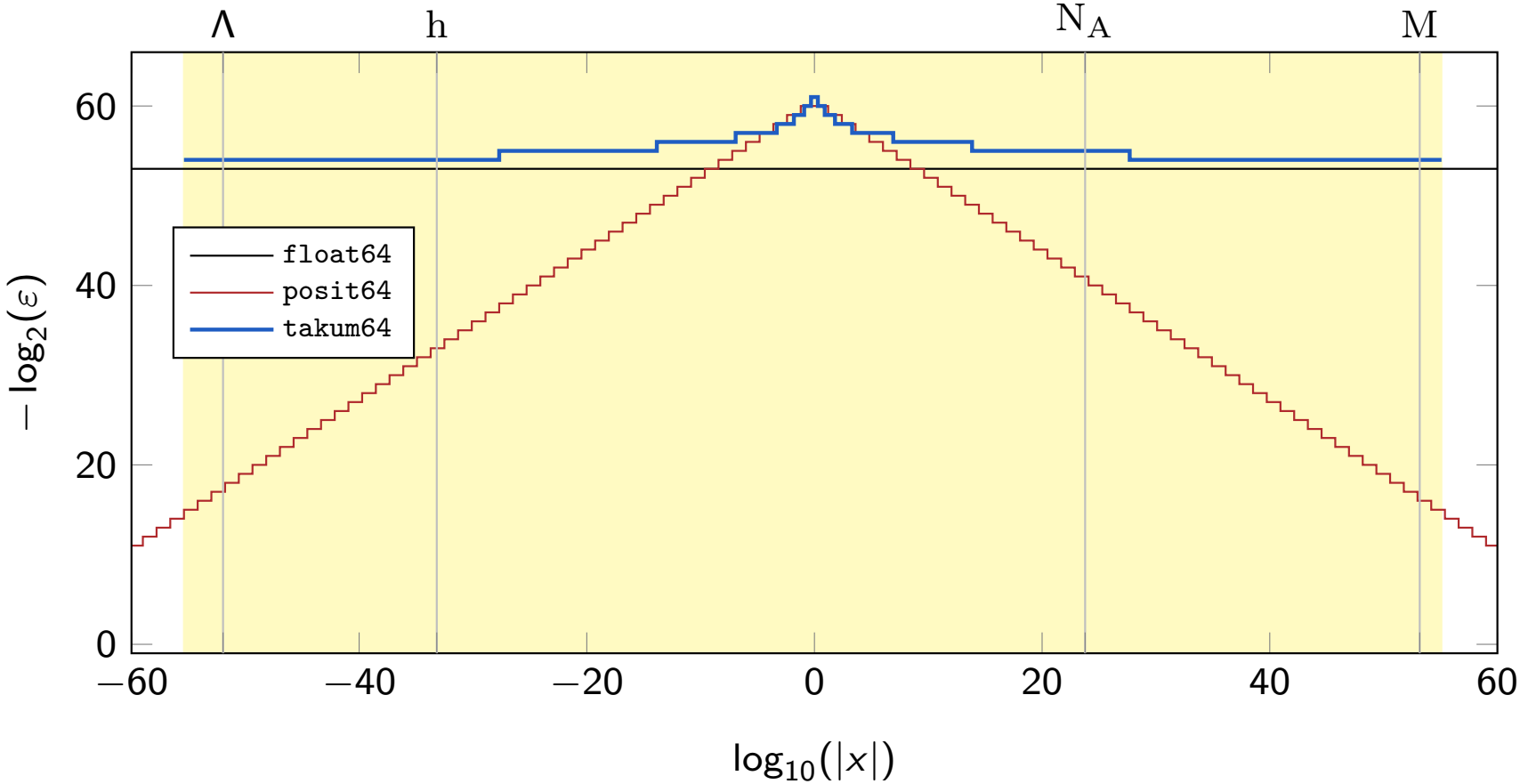


- Same golden zone as with $n = 16$
- Up to 32 times denser than `float32`

Takum

Density (64 Bit)



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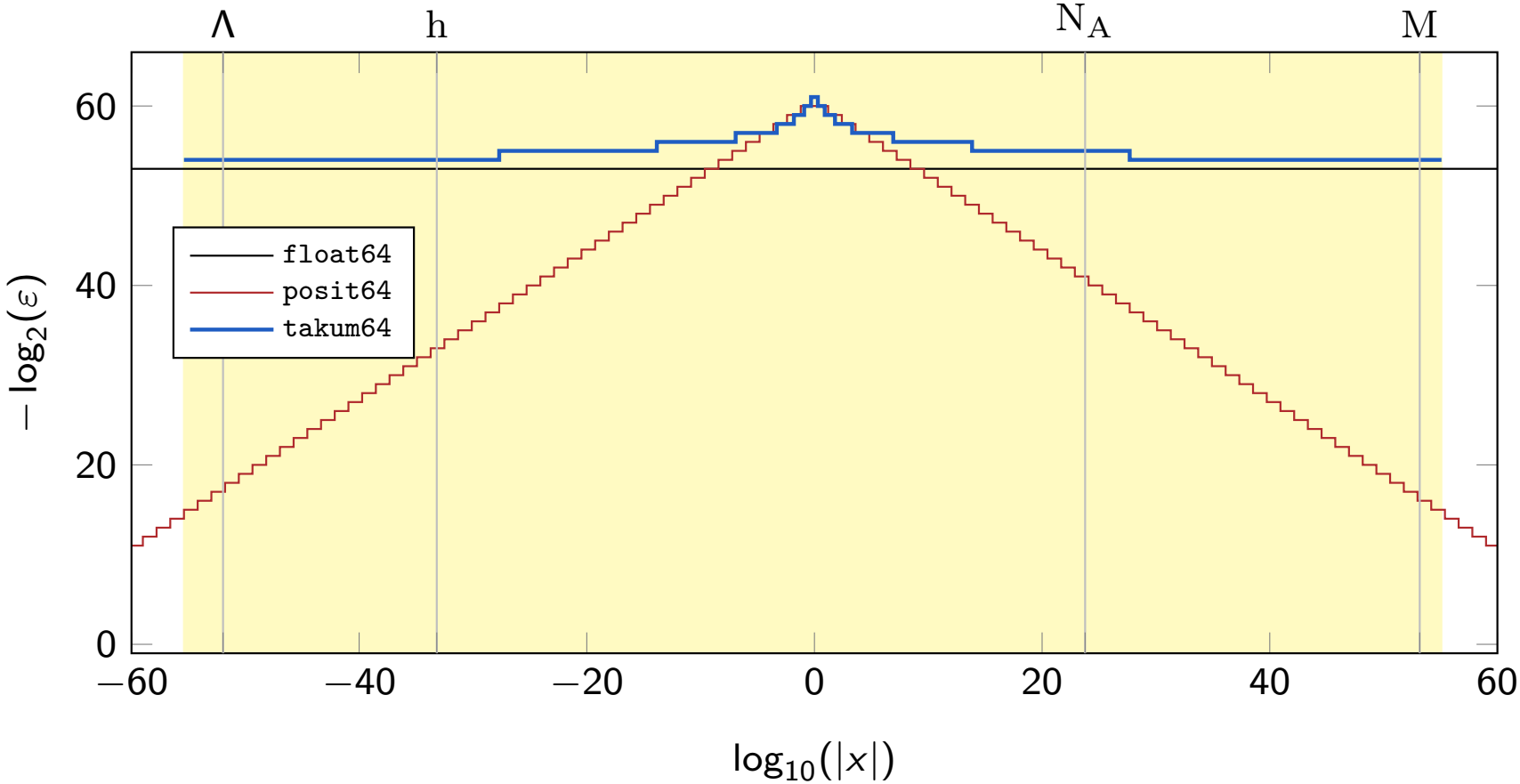

- At least 2 and up to 256 times denser than `float64`

Takum

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- Defined for all n, conversion between lengths simple rounding or expansion

FPGA Implementation (Artix 7 (100 MHz), VHDL)

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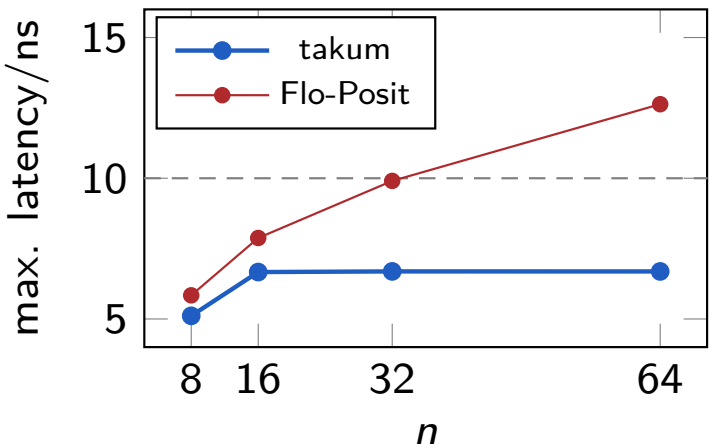

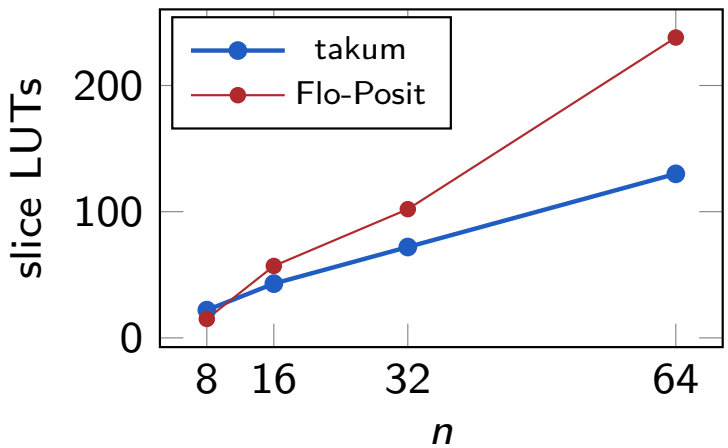

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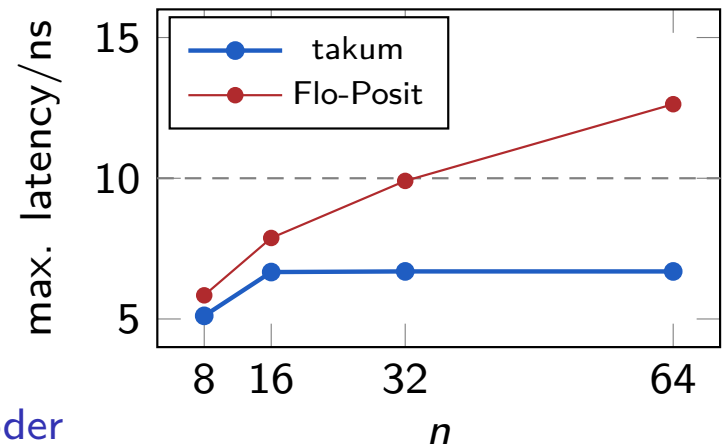

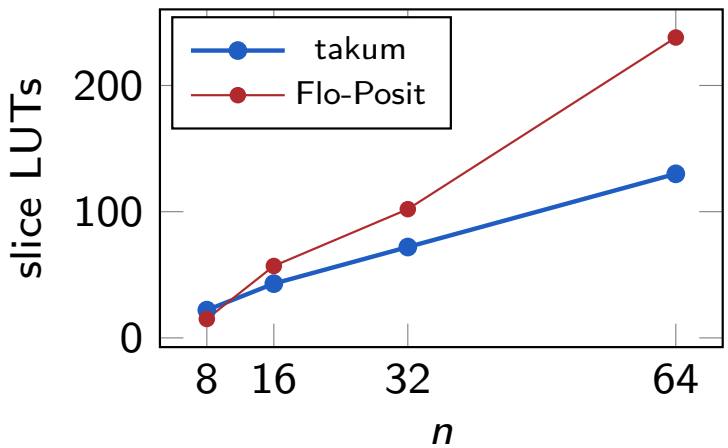

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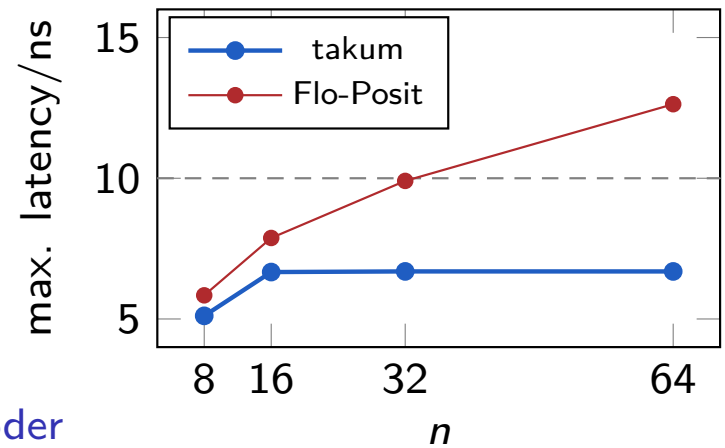

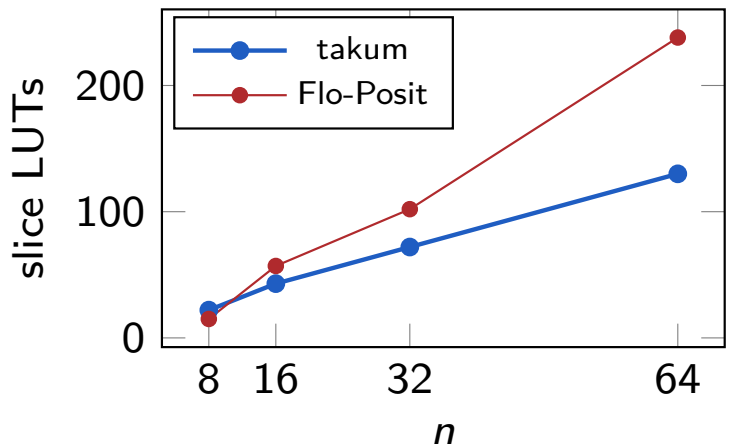

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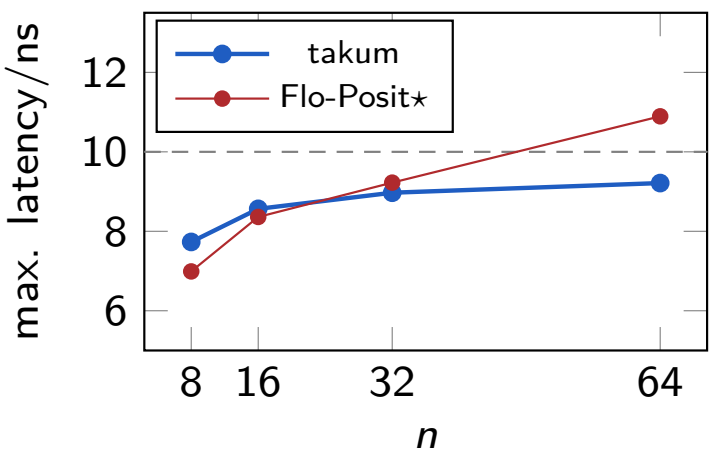

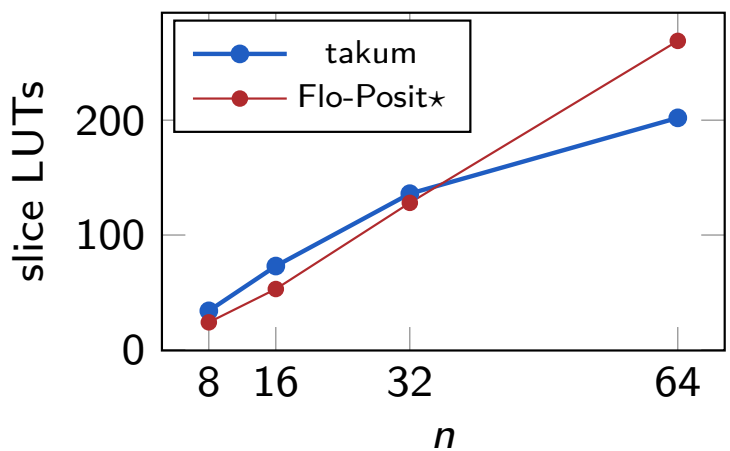

⋆: no under-/overflow detection

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Time for a new paradigm?


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arithmetic