

Takum: A New Tapered Precision Machine Number Format

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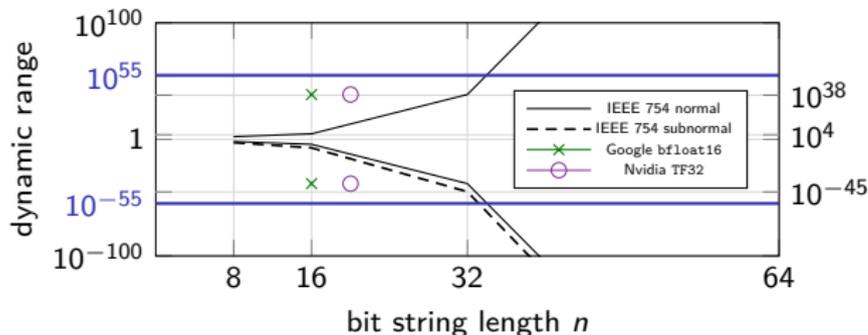


IEEE 754 Floating-Point



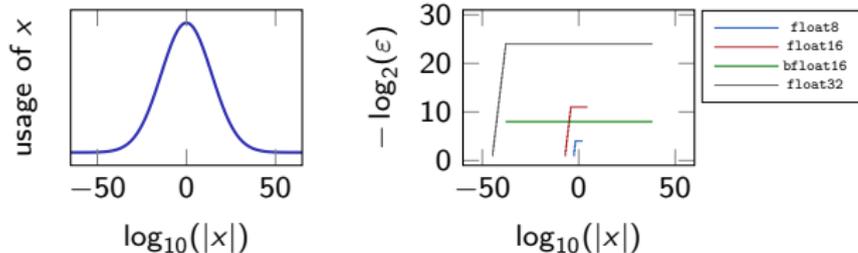
versus Desirable Design Criteria (1/2)

- ▶ Represented numbers should reflect those used in computations: Observations:
 - ▶ **Dynamic Range:** General purpose $10^{\pm 55}$ (SI units, constants, etc.)



Violated: too small for $n \leq 32$, too large for $n \geq 64$

- ▶ **Tapering:** Numbers in magnitude close to one are used much more frequently than very large or very small numbers



Violated: no tapering, uniform density (except subnormals)

IEEE 754 Floating-Point

versus Desirable Design Criteria (2/2)



- ▶ No redundant encodings

Violated: signed zero, redundant NaNs, excess numbers ($n \geq 64$)

format	float8	float16	bfloat16	TF32	float32	float64	float128
waste/%	5.08	3.12	0.78	0.78	0.39	82.03	98.88

- ▶ Defined for bit strings of any length

Violated: Only defined for $n \in \{8, 16, 32, 64, \dots\}$

- ▶ Straightforward conversion between lengths

Violated: Requires exponent-reencoding

- ▶ Simple/efficient hardware implementation (transistor count, energy efficiency, latency, throughput, etc.)

Violated: Special cases complicate hardware

- ▶ Subnormals (increase FPU energy consumption by 20%)
- ▶ Signed zero ($0 = -0$, $\sqrt{-0} = -0$, etc.)
- ▶ Signaling/quiet NaN ($\text{NaN} \neq \text{NaN}$, $\text{NaN}^0 = 1$, $1^{\text{NaN}} = 1$, etc.)
- ▶ Four rounding modes

IEEE 754 Floating-Point

Conclusion

End of 'double precision everywhere' era

- ▶ Deep learning/LLMs
- ▶ Mixed Precision
- ▶ Hardware/energy constraints (IoT, green HPC)

The standard is failing

- ▶ Not suitable for low precision applications ($n \leq 16$)
- ▶ Dilution with proprietary formats: bfloat16 and TF32, however still suffer from inherent issues
- ▶ A lot of new hardware (e.g. GPUs) openly violates the standard

The race is on

- ▶ IEEE 754 is a dead end with very high inertia
- ▶ Machine Numbers are the backbone of all computations, even small improvements have large high-level effects
- ▶ Time for a new paradigm?

Tapered Precision Machine Numbers

Motivation

- ▶ Consider general floating-point format

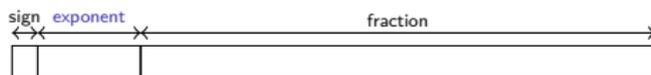


- ▶ Sign and fraction bits already contain maximum information

Angle of approach: exponent bits

- ▶ Want higher density of numbers (i.e. more fraction bits, density) for exponent values close to zero
- ▶ Solution: **variable-length exponent encoding**

- ▶ Small magnitude: shorter exponent, longer fraction, higher density



- ▶ Large magnitude: longer exponent, shorter fraction, lower density



Posit Arithmetic

Discussion

Tapering

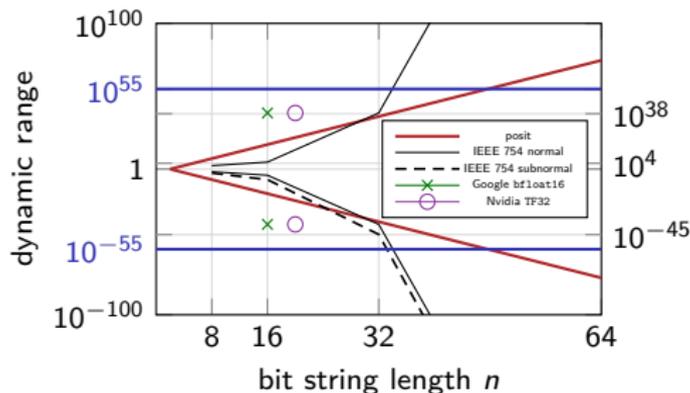
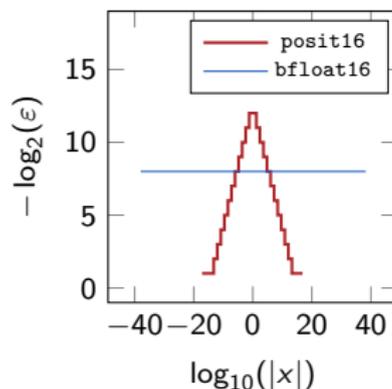
- ▶ Linear tapering
- ▶ Up to 16 times higher density for $n = 16$ compared to bfloat16

Dynamic Range

- ▶ Very limited dynamic range
- ▶ Rejected by Google in favour of bfloat16 for that reason

Conclusion

- ▶ interesting properties
- ▶ not for general purpose arithmetic



Logarithmic Number System

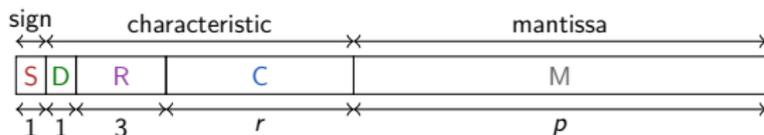
- ▶ Used internally at Meta for AI-ASICs (MTIA)
- ▶ Instead of $(-1)^s(1+f) \cdot 2^e$ express numbers as $(-1)^s b^{e+f} =: (-1)^s b^\ell$ for basis $b > 1$ (usually $b = 2$)
- ▶ ℓ is fixed-point number, exponent $e \rightarrow$ characteristic c , fraction $f \rightarrow$ mantissa m
- ▶ Advantages
 - ▶ Multiplication, Division, Square Root, Squaring, Inversion are simple fixed-point addition, subtraction, right and left shifts and negation, e.g.
$$b^{\ell_1} \cdot b^{\ell_2} = b^{\ell_1+\ell_2}, \sqrt{b^\ell} = b^{\ell/2}, (b^\ell)^{-1} = b^{-\ell}$$
 - ▶ Lower relative approximation error
 - ▶ Simpler hardware, more energy efficient, especially for AI²
- ▶ Disadvantages
 - ▶ Addition/subtraction more involved, but on-par with floating-point for $n \leq 32$ in regard to latency and area³
 - ▶ Binary zero representation challenge
- ▶ All previous work for $b = 2$, advances⁴ in mixed-base $(-1)^s 2^e e^f$ for $n > 32$

Define takum as LNS with pure base \sqrt{e} (transform to and from base e with ℓ -shift)

² Jeff Johnson. 'Rethinking floating point for deep learning'. arXiv: 1811.01721 (Nov. 2018)

³ J. Nicholas Coleman and Rizalafande Che Ismail. 'LNS with Co-Transformation Competes with Floating-Point'. In: IEEE Transactions on Computers 65.1 (2016), pp. 136-146. DOI: 10.1109/TC.2015.2409059

⁴ Jeff Johnson. 'Efficient, arbitrarily high precision hardware logarithmic arithmetic for linear algebra'. In: 2020 IEEE 27th Symposium on Computer Arithmetic (ARITH). Los Alamitos, CA, USA: IEEE Computer Society, June 2020, pp. 25-32. DOI: 10.1109/ARITH48897.2020.00013



Sign bit S , direction bit D , regime bits R , characteristic bits C , mantissa bits M

$$r := \begin{cases} \text{uint}(\bar{R}) & D = 0 \\ \text{uint}(R) & D = 1 \end{cases} \in \{0, \dots, 7\} \quad : \text{regime}$$

$$c := \begin{cases} -2^{r+1} + 1 + \text{uint}(C) & D = 0 \\ 2^r - 1 + \text{uint}(C) & D = 1 \end{cases} \quad : \text{characteristic}$$

$$p := n - r - 5 \in \{n - 12, \dots, n - 5\} \quad : \text{mantissa bit count}$$

$$m := 2^{-p} \text{uint}(M) \in [0, 1) \quad : \text{mantissa}$$

$$\ell := (-1)^S (c + m) \in (-255, 255) \quad : \text{logarithmic value}$$

$$\tau((S, D, R, C, M)) := \begin{cases} \begin{cases} 0 & S = 0 \\ \text{NaN} & S = 1 \end{cases} & D = R = C = M = 0 \\ (-1)^S \sqrt{e}^\ell & \text{otherwise} \end{cases}$$

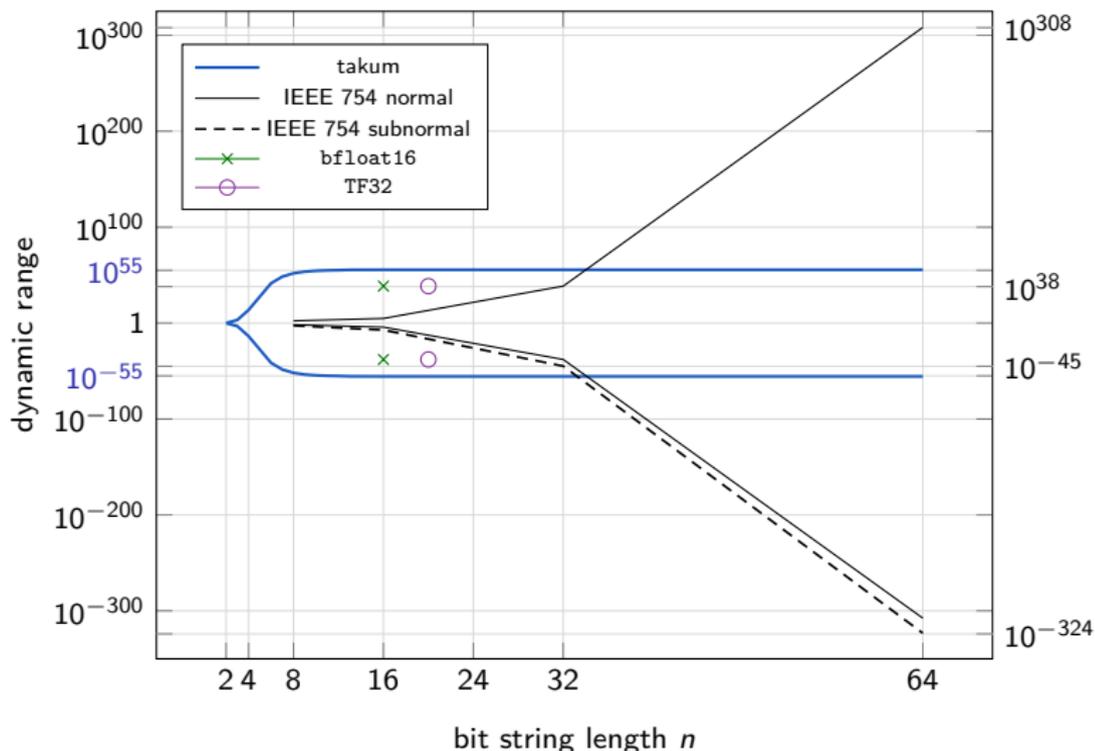
Decoding

- ▶ Only 16 distinct cases via (D, R) , just two LUTs (32 byte, 16 byte)
- ▶ (D, R, C) is at most 11 bits long \rightarrow shared logic for all n
- ▶ Computation of c is conditional negation, bitwise OR and increment

Takum

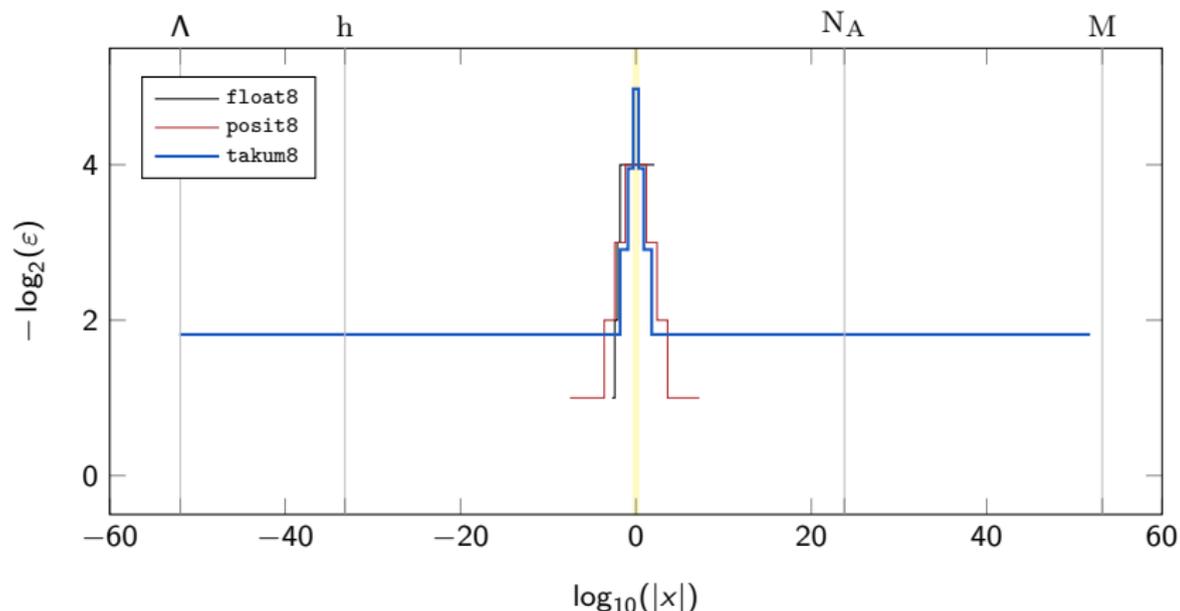
Dynamic Range

- ▶ Dynamic range $(\sqrt{e}^{-255}, \sqrt{e}^{255}) \approx (4.2 \times 10^{-56}, 2.4 \times 10^{55})$ for $n \geq 12$



Takum

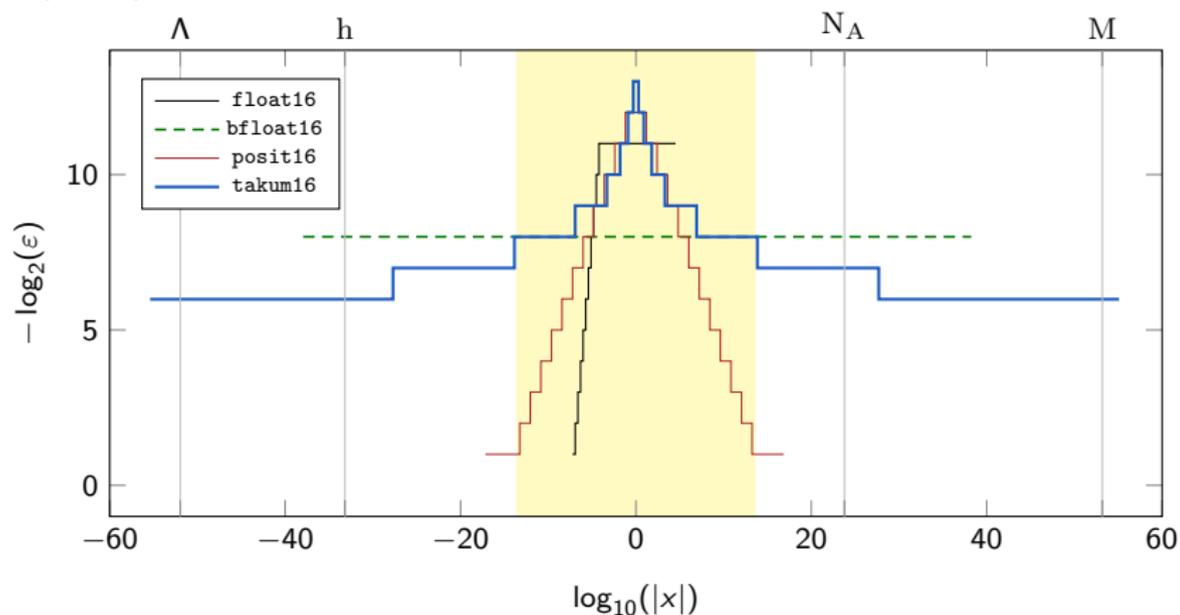
Density (8 Bit)



- ▶ takum8 exceeds float8 ('golden zone') and posit8 near unity (2 times denser)
- ▶ Almost full dynamic range across benchmark constants Λ (cosmological constant), h (PLANCK constant), N_A (AVOGADRO constant), M (mass of the universe)

Takum

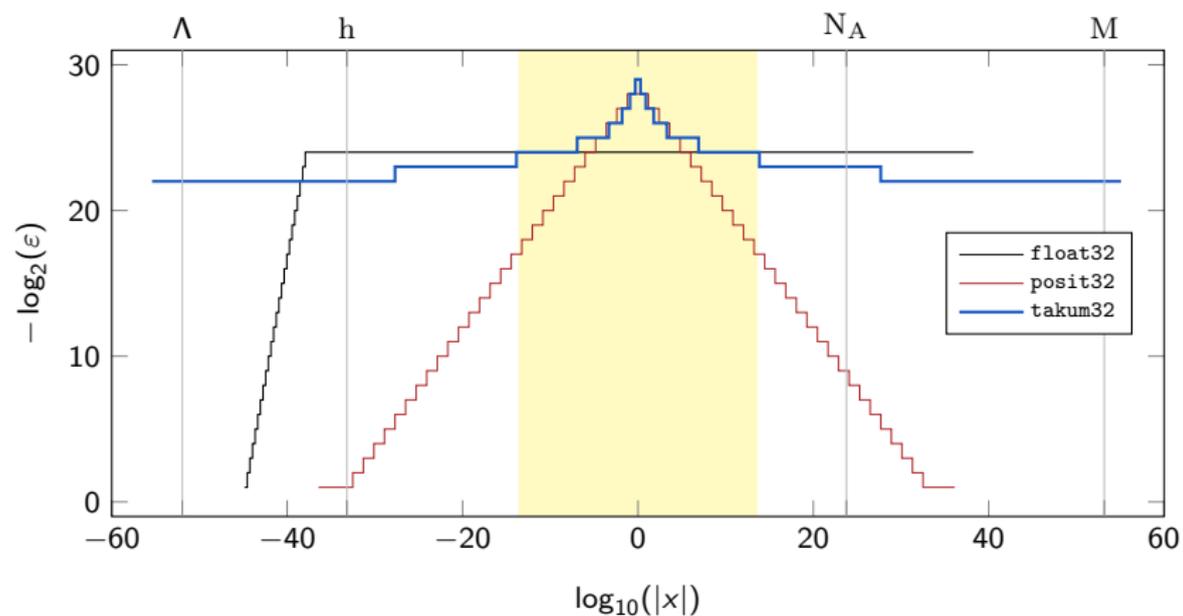
Density (16 Bit)



- ▶ Perfect logarithmic tapering
- ▶ At least the same density as bfloat16 in 'golden zone' ($10^{-17}, 10^{17}$), up to 32 times denser
- ▶ Up to 2 times denser than posit16

Takum

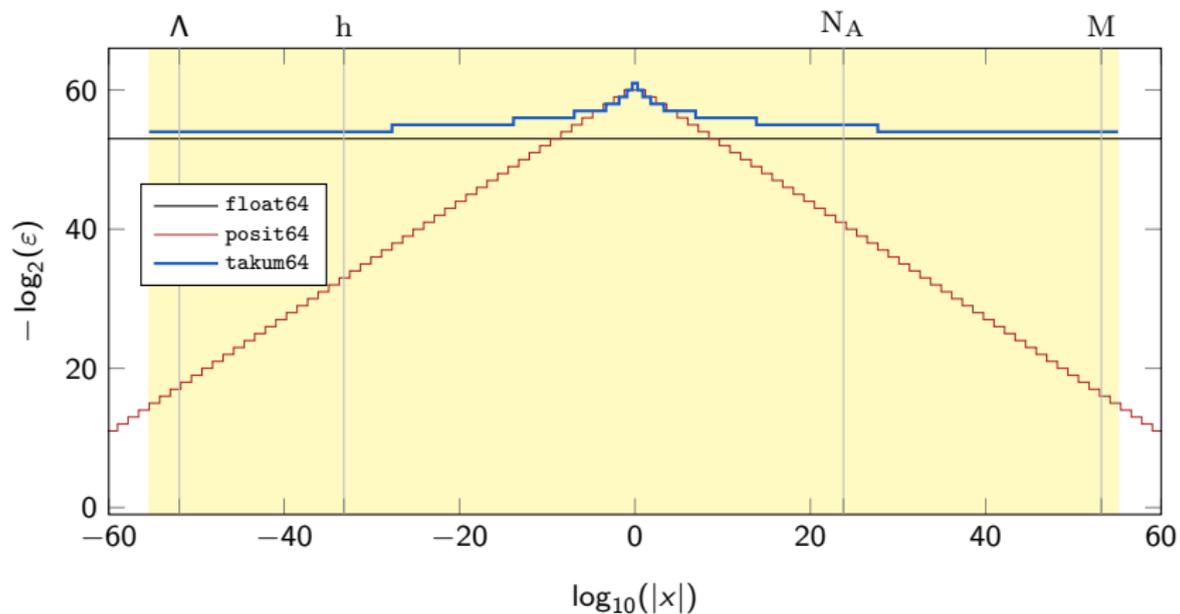
Density (32 Bit)



- ▶ Same golden zone as with $n = 16$
- ▶ Up to 32 times denser than float32

Takum

Density (64 Bit)



- ▶ At least 2 and up to 256 times denser than float64

Takum

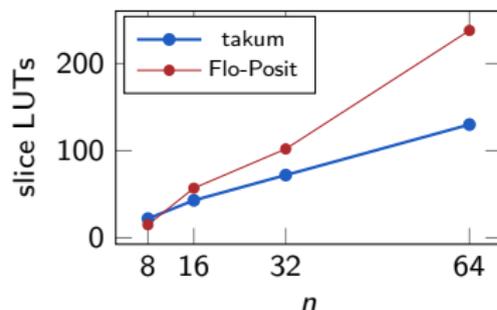
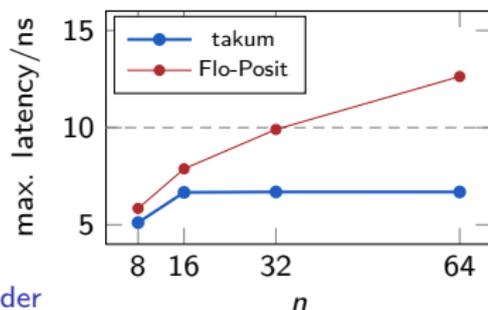
Further Properties

- ▶ No redundant representations
- ▶ $\tau(\mathbf{0}) = 0$ (unique, unsigned zero)
- ▶ $\tau(-B) = -\tau(B)$ (two's complement negation, no extra hardware needed)
- ▶ $B \preceq \tilde{B} \implies \tau(B) \leq \tau(\tilde{B})$ (total two's complement order, also for NaN (!), no extra hardware needed)
- ▶ Inversion is a simple bitwise operation
- ▶ Defined for all n , conversion between lengths simple rounding or expansion

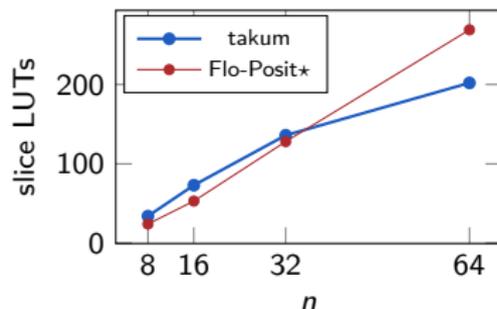
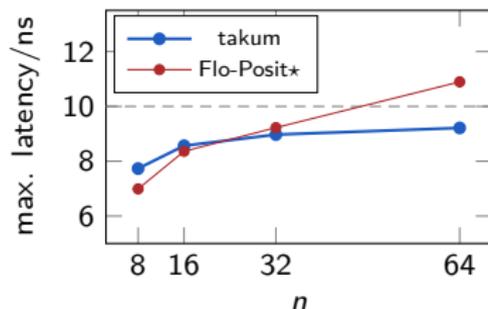
FPGA Implementation (Artix 7 (100 MHz), VHDL)

Comparison against the state of the art (FloPoCo with improvements, 2018-2023)⁵

Decoder



Encoder



*: no under-/overflow detection

⁵ Raul Murillo et al. 'Comparing different decodings for posit arithmetic'. Conference on Next Generation Arithmetic, 2022

Conclusion and Outlook

- ▶ Takums promising for low-precision (e.g. AI) *and* general purpose arithmetic
- ▶ True mixed-precision workflow
- ▶ Novel base \sqrt{e} LNS approach for simpler and more power efficient hardware
- ▶ Takum FPGA implementation superior to fastest FPGA posit codec
- ▶ C99 Implementation: <https://github.com/takum-arithmetic/libtakum>
- ▶ Future work: Further evaluation (deep learning, numerical methods, etc.), ASIC implementation, 'tapered precision numerical analysis'
- ▶ <https://takum.org>
- ▶ Full paper: 'Beating Posits at Their Own Game: Takum Arithmetic'. In: *Conference on Next Generation Arithmetic (CoNGA) 2024*. Lecture Notes in Computer Science. In publication. Berlin, Heidelberg: Springer Berlin Heidelberg, 2024
<https://arxiv.org/abs/2404.18603>

IEEE 754 is a dead end.

Even tiny machine number improvements yield large high-level effects.

Time for a new paradigm?

